

# Constant mean curvature surfaces of Delaunay type along a closed geodesic

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## Abstract

In this paper, we construct Delaunay type constant mean curvature surfaces along a non-degenerate closed geodesic in a 3-dimensional Riemannian manifold.

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## 1 Introduction

Constant mean curvature (CMC) surfaces are a class of important submanifolds. Let  $(M^{m+1}, g)$  be a Riemannian manifold of  $m + 1$  dimension. We consider the embedded CMC hypersurfaces. In early 1990s, R. Ye proved in [17] the existence of the foliation of constant mean curvature spheres in Riemannian manifolds around the non-degenerate critical points of the scalar curvature. In 1996, in [5] G. Huisken and S. T. Yau proved the existence of constant mean curvature foliation in the asymptotically flat end (of a manifold) with positive mass. Huisken and Yau's result were extended by Lan-hsuan Huang in [4] and C. Nerz in [11]. Similar problems were also considered in asymptotically hyperbolic manifolds. For this topic, see [15, 12, 13, 10]. In [14], Pacard and Xu proved the existence of constant mean curvature spheres around the degenerate critical points of scalar curvature which can be regarded as another extension of Ye's result. In [9], Mazzeo and Pacard proved the existence of constant mean curvature tubes along a closed non-degenerate geodesic. Geodesic is a kind of simple minimal submanifold. In [7], Mahmoudi, Mazzeo and Pacard proved the existence of constant mean curvature hypersurfaces along minimal submanifolds. In contrast to the result of Rugang Ye in [17], the CMC hypersurfaces constructed in [9] and [7] constitute a partial foliation, that is a foliation with gaps. There is a good reason for such gaps. Around each gap, bifurcation occurs. The CMC surfaces which bifurcate from the tubes are of Delaunay type.

We will give the definition of Delaunay surfaces in the next section. The Delaunay surfaces were discovered in 1841 by C. Delaunay in [2]. It is a one parameter family of complete non-compact surfaces in  $\mathbb{R}^3$ . The Delaunay surfaces have rotational symmetry. So the surfaces are decided by an ODE. One may refer to [3] for a description of Delaunay surfaces. The Delaunay surfaces play an analogous role in the theory of complete CMC surfaces as catenoids do in the theory of complete minimal surfaces. In [16], it is proved that any complete minimal immersion  $M^n \subset \mathbb{R}^{n+1}$  with two embedded ends and with finite total curvature must be a catenoid or a pair of planes. And it is proved in [6] that any CMC surface embedded in  $\mathbb{R}^3$  having two ends is a Delaunay surface. Another

fact is that any end of a complete minimal surface of finite total curvature must be asymptotic to a catenoid or a plane. Paralleling to this fact, each end of an embedded CMC surface with finite topology converges exponentially to the end of some Delaunay surface. The Delaunay type hypersurfaces exist in  $\mathbb{S}^n$ ,  $\mathbb{R}^n$  and  $\mathbb{H}^n$ . Note that those embedded in  $\mathbb{S}^n$  are compact which bifurcate from CMC tori. In the more recent work [1], Bettiol and Piccione managed to construct Delaunay type CMC surfaces in cohomogeneity one manifolds. Cohomogeneity one manifolds are those support an isometric action of a Lie group such that the orbit space  $M/G$  is one dimensional. We see that the metric on cohomogeneity one manifolds are not generic. For generic metrics, the existence of Delaunay type CMC surfaces is unknown, despite some partial results. In this paper we focus on the existence of Delaunay type CMC surfaces along closed geodesics in generic metrics, where the geodesics can be assumed to be non-degenerate. As mentioned just now, this kind of surfaces can be regarded as the bifurcation branches of the CMC tubes constructed in [9]. One can refer to [9] for a description of the moduli space of CMC surfaces along  $\Gamma$  which are isotopic to geodesic tubes, which is the motivation of this paper.

Let's state the main theorem roughly:

**Theorem 1.1.** *Suppose  $(M^3, g)$  is a 3-dimensional Riemannian manifold and  $\Gamma$  is a closed embedded geodesic with non-degenerate Jacobi operator. Then for any  $\tau_0 \in (0, \frac{1}{4})$  we can find  $\varepsilon_0 > 0$  which depends on the manifold  $M$  and  $\tau_0$  such that there is a monotone sequence  $\varepsilon_n \rightarrow 0$  with  $\varepsilon_0 > \varepsilon_1 > \dots > \varepsilon_n > \dots$  such that along the geodesic there are at least two embedded constant mean curvature surfaces of Delaunay type, with mean curvature  $2/\varepsilon_n$  and of size  $\varepsilon_n$ . Moreover, the Delaunay parameter of the surfaces is close to  $\tau_0$ .*

The terms “Delaunay type”, “size  $\varepsilon_n$ ” and “the parameter is close to  $\tau_0$ ” will be made clear in Theorem 2.3 which is the rigorous statement of this theorem.

The assumption that the Jacobi operator is non-degenerate is a mild restriction which holds for generic metrics. The condition that  $\Gamma$  is embedded is not essential. If  $\Gamma$  is immersed, the construction also works. However, the Delaunay type surfaces are also immersed in this case.

Our method can also be used in the case that there is symmetry.

**Corollary 1.2.** *If the metric has rotational symmetry (at least in a tubular neighborhood of  $\Gamma$ ) with respect to the closed geodesic  $\Gamma$ , then we can remove the condition that the Jacobi operator of the geodesic is non-degenerate in Theorem 1.1.*

In [7], the authors also did similar things as in [9], namely, if  $\Sigma \subset M$  is a compact non-degenerate closed minimal submanifold and  $\Sigma$  is at least of codimension two, then the authors constructed a partial foliation of CMC surfaces condensing along  $\Sigma$ . The authors believe that there should be also bifurcation phenomenon in this setting. However geometric picture of the bifurcating CMC surfaces is still not clear in this case. So actually our work is only the first step in this area. There are many things left to be understood.

We use the perturbation method to solve this problem. This paper is organized as follows: In Section 2, we revise some basic facts of Delaunay unduloid (which

we call Delaunay surface) embedded in the Euclidean space  $\mathbb{R}^3$  and we make a description on how we arrange an initial surface along the closed geodesic and how we perturb the initial surface. Then after long calculations we get the mean curvature of the perturbed initial surface. In Section 3 we analyze the Jacobi operator of the perturbed initial surface. We divide the function space into 3 parts according to the invariant subspaces of the Jacobi operator. When restricting the Jacobi operator to each part we have high mode, 1st mode and 0th mode. For high mode, it is easy to prove that the operator is invertible and the inverse has good bounds. For 1st mode, after careful examination, we find the Jacobi operator converges in certain sense to the Jacobi operator of the geodesic, which is invertible by the assumption that the geodesic is non-degenerate. Here we prove an “average 1” lemma which verifies this convergence. For 0th mode, if we solve the linearized equation directly, the solution would be too big to carry out the fixed point theorem. First we solve a nonlinear ODE (20) which is related to prescribed mean curvature surfaces and we can get some basic estimates for the solution. Then we analyze its linearizations when the prescribed terms  $\xi, \mu, \omega, \phi(0)$  have variations. Also we discuss how to choose proper parameters  $\omega, \phi(0)$  to get global smooth solutions to the nonlinear ODE. At the end of 0th mode, we get main estimates (23), which tells us how the global smooth surfaces depend on two prescribed terms  $\xi, \mu$ . An “average 0” lemma is important in 0th mode. In section 4, we use fixed point theorem to solve the 3 modes together. Note that the linearized equation of 0th mode has kernel (maybe the operator is invertible but the inverse is too big). Actually this kernel term  $\phi_\psi$  appears in the prescribed term of the nonlinear ODE. So the mean curvature of the surface we get is constant plus the kernel of 0th mode. At last, by analyzing the energy functional we manage to remove the kernel term and find at least two constant mean curvature surfaces of Delaunay type.

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## 2 Geometry of Delaunay surfaces

### 2.1 The initial surface and the perturbation

#### 2.1.1 Delaunay surfaces in Euclidean space $\mathbb{R}^3$

First we give a brief revision of the definition of Delaunay surfaces in Euclidean space  $\mathbb{R}^3$ . There are two kinds of Delaunay surfaces in  $\mathbb{R}^3$ , Delaunay nodoids and Delaunay unduloids. The first type can be immersed into  $\mathbb{R}^3$ , and the second type can be embedded into  $\mathbb{R}^3$ . In this paper, by Delaunay surface, we always mean

Delaunay unduloid. The Delaunay unduloid  $D_\tau$  can be parameterized by

$$X_\tau(s, \theta) := (\phi_\tau(s) \cos \theta, \phi_\tau(s) \sin \theta, \psi_\tau(s)),$$

where  $(s, \theta) \in \mathbb{R} \times S^1$ . Here  $\tau$  is a real parameter and  $0 < \tau \leq \frac{1}{4}$ .  $(\phi, \psi)$  is the solution to the following system

$$\begin{cases} \dot{\phi}^2 + (\phi^2 + \tau)^2 = \phi^2, & \phi(0) = \frac{1 - \sqrt{1 - 4\tau}}{2} \\ \dot{\psi} = \phi^2 + \tau, & \psi(0) = 0, \end{cases} \quad (1)$$

where the derivative “.” is taken with respect to parameter  $s$ . This solution is periodic.

*Remark.* This solution to (1) is periodic and satisfies

$$\frac{1 - \sqrt{1 - 4\tau}}{2} \leq \phi(s) \leq \frac{1 + \sqrt{1 - 4\tau}}{2}.$$

So  $\phi(s)$  attains its minimum at  $s = 0$ .

From  $\dot{\psi} = \phi^2 + \tau$  we know  $\psi$  is a strictly increasing function of  $s$ . So we can also regard  $\phi$  as a function of  $\psi$ . Then  $\phi(\psi)$  has to satisfy a second order ODE. This ODE can also be regarded as another definition of Delaunay surfaces. From

$$\dot{\phi}^2 + (\phi^2 + \tau)^2 = \phi^2,$$

we have

$$\ddot{\phi} = \phi - 2\phi(\phi^2 + \tau).$$

Also we know  $\dot{\phi} = \phi'(\psi)\dot{\psi}$  and  $\ddot{\phi} = \phi_{\psi\psi}(\psi)(\phi^2 + \tau)^2 + 2\phi\phi_{\psi}^2(\phi^2 + \tau)$ , where “'” or “ $\phi_{\psi}$ ” denote the derivative with respect to  $\psi$ . At last, we have

$$\phi^2 + \tau = \frac{\phi}{\sqrt{1 + \phi_{\psi}^2}}.$$

Combining all these,  $\phi(\psi)$  satisfies

$$\begin{cases} \phi_{\psi\psi} - \phi^{-1}(1 + \phi_{\psi}^2) + 2(1 + \phi_{\psi}^2)^{\frac{3}{2}} = 0, \\ \phi(0) = \frac{1 - \sqrt{1 - 4\tau}}{2}, \\ \phi_{\psi}(0) = 0. \end{cases} \quad (2)$$

We can also find different definitions of Delaunay surfaces in [8]. By direct calculations, we know  $X_\tau(s, \theta) = (\phi \cos \theta, \phi \sin \theta, \psi) \subset (\mathbb{R}^3, d\psi^2 + d\phi^2 + \phi^2 d\theta^2)$  has mean curvature

$$-\phi_{\psi\psi}(1 + \phi_{\psi}^2)^{-3/2} + \phi^{-1}(1 + \phi_{\psi}^2)^{-1/2}.$$

From this and the ODE (2), we get the mean curvature of Delaunay surfaces is equal to 2 independent of  $\tau \in (0, \frac{1}{4}]$ . Actually, when  $\tau = \frac{1}{4}$ , we get a limit solution

to ODE (2),  $\phi \equiv \frac{1}{2}$  which defines a cylinder of radius  $\frac{1}{2}$ . If  $\tau \rightarrow 0$ , the solutions of ODE (2) tend to a limit function  $\phi = \sqrt{1 - (\psi - 2i - 1)^2}$ ,  $i \in \mathbb{Z}$ . This defines a singular limit of Delaunay surfaces which is infinitely many spheres of radius 1 with each sphere meeting its two neighbors at the two poles.

The reason why the system has only periodic solutions is that it has a first integral

$$\tau(\phi, \phi_\psi) = -\phi^2 + \frac{\phi}{\sqrt{1 + \phi_\psi^2}}.$$

It is easy to check that  $\tau$  keeps constant along the solution to (1) or (2).

### 2.1.2 Fermi coordinates and Taylor expansion of the metric near the geodesic

From now on we discuss the geometry of Delaunay surface along a geodesic in a Riemannian 3-manifold. Fix an arc length parametrization  $x_0$  of the geodesic  $\Gamma$ ,  $x_0 \in [0, L_\Gamma]$ , where  $L_\Gamma$  is the length of  $\Gamma$ . We denote the normal bundle of  $\Gamma$  by  $N\Gamma$ . Choose a parallel orthonormal basis  $E_1, E_2$  for  $N\Gamma$  along  $[a, b]$ . This determines a coordinate system

$$x : (x_0, x_1, x_2) \mapsto \exp_{\Gamma(x_0)}(x_1 E_1 + x_2 E_2) := F(x),$$

and we denote the corresponding coordinate vector fields by  $X_\alpha := F_*(\partial_{x_\alpha})$ . We adopt the convention that indices  $i, j, k, \dots \in \{1, 2\}$  while  $\alpha, \beta, \dots \in \{0, 1, 2\}$ . Let  $r = \sqrt{x_1^2 + x_2^2}$ . By Gauss' Lemma  $r$  is the geodesic distance from  $x$  to  $\Gamma$  and the vector  $\partial_r = \frac{1}{r}(x_1 X_1 + x_2 X_2)$  is perpendicular to  $X_0$ . We also know  $\partial_\theta = -x_2 X_1 + x_1 X_2$ .

It is easy to see that the metric coefficients  $g_{\alpha\beta} = \langle X_\alpha, X_\beta \rangle$  equal  $\delta_{\alpha\beta}$  along  $\Gamma$ . Now we are going to calculate higher order terms in the Taylor expansions of  $g_{\alpha\beta}$ . By the notation  $O(r^m)$ , we mean a function  $f$  such that it and its partial derivatives of any order, with respect to the vector fields  $X_0$  and  $x_i X_j$ , are bounded by  $C r^m$  in some fixed  $T_{\rho_0}(\Gamma) = \{p | r(p, \Gamma) \leq \rho_0\}$ .

First for the covariant derivative, we have

**Lemma 2.1.** *For  $\alpha, \beta = 0, 1, 2$ ,*

$$\nabla_{X_\alpha} X_\beta = \sum_{\gamma=0}^2 O(r) X_\gamma,$$

*and more precisely for  $\alpha = \beta = 0$ , we have*

$$\nabla_{X_0} X_0 = - \sum_{i,j=1}^2 R(X_j, X_0, X_i, X_0)_p x_i X_j + \sum_{\gamma=0}^2 O(r^2) X_\gamma,$$

*where  $R(X_i, X_j, X_k, X_l) = \langle \nabla_{X_i} \nabla_{X_j} X_k - \nabla_{X_j} \nabla_{X_i} X_k - \nabla_{[X_i, X_j]} X_k, X_l \rangle$ .*

*Proof.* We follow Lemma 2.1 in [9]. At any point  $p \in \Gamma$

$$\nabla_{X_0} X_0 = \nabla_{X_0} X_j = \nabla_{X_j} X_0 = \nabla_{X_i} X_j = 0$$

where the last one holds because on  $\Gamma$ ,  $\nabla_{X_i} X_i = 0$ ,  $\nabla_{X_i+X_j}(X_i+X_j) = 0$ . So the first equality follows. For the second one note that

$$\begin{aligned} X_i < \nabla_{X_0} X_0, X_j >_p &= < \nabla_{X_i} \nabla_{X_0} X_0, X_j >_p + < \nabla_{X_0} X_0, \nabla_{X_i} X_j >_p \\ &= < \nabla_{X_i} \nabla_{X_0} X_0, X_j >_p + O(r^2) \\ &= < R(X_i, X_0) X_0, X_j >_p + < \nabla_{X_0} \nabla_{X_i} X_0, X_j >_p + O(r^2) \\ &= < R(X_i, X_0) X_0, X_j >_p + O(r) \end{aligned}$$

which implies the second one.  $\square$

The next lemma gives the expansion of the metric coefficients in Fermi coordinates.

**Lemma 2.2.** *In the same notation as before, we have*

$$\begin{aligned} g_{ij}(q) &= \delta_{ij} + \frac{1}{3} R(X_k, X_i, X_l, X_j)_p x_k x_l + O(r^3) \\ g_{0i}(q) &= \frac{2}{3} R(X_k, X_0, X_l, X_i)_p x_k x_l + O(r^3) \\ g_{00}(q) &= 1 + R(X_k, X_0, X_l, X_0)_p x_k x_l + O(r^3) \end{aligned} \tag{3}$$

*Proof.* The reader may refer to the proof of Proposition 2.1 in [9] for the proof. However, we give more accurate expansion for  $g_{0i}$ .  $\square$

### 2.1.3 Initial Delaunay surfaces and the perturbation

First we arrange an initial Delaunay surface of size  $\varepsilon$  along the geodesic. Choose a starting point  $p_0$  with  $x_0(p_0) = 0 \pmod{L_\Gamma}$  on the closed geodesic  $\Gamma$ . We fixed a parameter  $\tau_0 \in (0, 1/4)$ . We assume  $\phi_{\tau_0}(\psi)$  is the solution to ODE (2) with  $\tau = \tau_0$ . Suppose the period of  $\phi(\psi)$  is  $\psi_1(\tau_0)$ . Suppose  $\varepsilon$  is a small number such that  $L_\Gamma = \varepsilon \psi_1(\tau_0) N$ , where  $N \in \mathbb{N}^+$ . It is obvious that one can only choose a sequence of such  $\varepsilon$  which tends to 0. We call this sequence the proper sizes for  $\tau_0$  and  $L_\Gamma$ . With such  $\varepsilon$ , we can arrange a Delaunay type initial surface around the geodesic. Let's make it precise.

The unit circle bundle is locally trivialized by the map

$$[a, b] \times S^1 \ni (x_0, \Upsilon) \mapsto (\Gamma(x_0), \sum_{j=1}^2 \Upsilon_j E_j) \in S N \Gamma.$$

The image

$$F(x_0, \varepsilon \phi_{\tau_0}(\frac{x_0}{\varepsilon}) \Upsilon)$$

can be defined locally and extended globally when  $\varepsilon$  is a proper size for  $\tau_0$  and  $L_\Gamma$ , as it does not depend on the choice of orthonormal basis  $E_i$ . We denote this image by  $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}$ . The mean curvature  $H(\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon})$  is nearly  $\frac{2}{\varepsilon}$ . Our aim is to perturb this initial surface such that it has exactly constant mean curvature  $\frac{2}{\varepsilon}$ .

Consider the following perturbation of  $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}$ , denoted by  $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)$ , where  $w$  is a function on unit circle bundle  $SNT$  and  $\eta$  is a section of  $N\Gamma$ .

Fix  $\varepsilon > 0$ , and denote the image

$$F(x_0, \varepsilon(\phi(\frac{x_0}{\varepsilon}) + w(\frac{x_0}{\varepsilon}, \theta))\Upsilon + \eta(x_0));$$

by  $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)$ . It is obtained by first taking the vertical graph of the function  $\varepsilon w$  over the initial Delaunay surface  $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}$  and then translating it by  $\eta$ .

The spaces we will work in are the Hölder spaces.  $C_{x_0}^{m, \alpha}$  represents  $C_{x_0}^{m, \alpha}(\Gamma)$ ,  $C_{x_0}^{m, \alpha}(N\Gamma)$  or  $C_{x_0}^{m, \alpha}(SNT)$  which are base on the differentiations with respect to  $\frac{\partial}{\partial x_0}$  and  $\frac{\partial}{\partial \theta}$ , where  $\theta$  is the coordinates on  $S^1$ . Also we will use modified Hölder spaces  $C_\varepsilon^{m, \alpha}(\Gamma)$ ,  $C_\varepsilon^{m, \alpha}(N\Gamma)$  or  $C_\varepsilon^{m, \alpha}(SNT)$  which are based on differentiations with respect to the vector fields  $\varepsilon \partial_{x_0} = \frac{\partial}{\partial \psi}$  and  $\partial_\theta$ . Sometimes we omit the symbols  $\Gamma, N\Gamma, SNT$  when it is clear in the context. By  $C_{x_0, \varepsilon}^{2, \alpha}, C_{x_0, \varepsilon}^{1, \alpha}$  we mean the modified Hölder spaces with the following norm

$$\begin{aligned} \|f\|_{C_{x_0, \varepsilon}^{2, \alpha}} &= \|f\|_{C_{x_0}^1} + \varepsilon \left\| \frac{\partial^2 f}{\partial x_0^2} \right\|_{C^0} + \varepsilon^{1+\alpha} \left[ \frac{\partial^2 f}{\partial x_0^2} \right]_{C_{x_0}^\alpha}, \\ \|f\|_{C_{x_0, \varepsilon}^{1, \alpha}} &= \|f\|_{C_{x_0}^1} + \varepsilon^\alpha \left[ \frac{\partial f}{\partial x_0} \right]_{C_{x_0}^\alpha}. \end{aligned}$$

By  $\|(f, g)\|_{\varepsilon, \alpha}$  we mean

$$\|(f, g)\|_{\varepsilon, \alpha} = \|f\|_{C_{x_0}^1} + \varepsilon \|g\|_{C_\varepsilon^\alpha}.$$

For  $p \in \Gamma$ , let  $S_p^1$  denote the unit circle fibre of  $SNT$  over  $p$ . Any function  $w$  on  $SNT$  decomposes into a sum of three terms

$$w = w_0 + w_1 + \tilde{w}, \tag{4}$$

where the restriction to any  $S_p^1$  of each of these terms lies in the span of the eigenfunctions  $\xi_j$  on  $S^1$  with  $j = 0, j = 1, 2$ , and  $j > 2$ , respectively.  $w_0$  is a function on  $\Gamma$ .

$$\begin{aligned} w_1(s, \theta) &= w_1^1(s)\xi_1 + w_1^2(s)\xi_2 \\ &= w_1^1(s)\cos\theta + w_1^2(s)\sin\theta. \end{aligned}$$

Note that any linear combination of  $\xi_1$  and  $\xi_2$  can be identified with a translation in  $\mathbb{R}^2$  ( $\xi_1$  and  $\xi_2$  correspond to the translations in  $x$  and  $y$  direction). Correspondingly,  $w_1$  is canonically associated to a section  $\eta$  of the normal bundle  $N\Gamma$ .

At last

$$\tilde{w}(s, \theta) = \sum_{j \geq 2} \tilde{w}_j(s)\xi_j.$$



We denote by  $\Pi_0$ ,  $\Pi_1$  and  $\tilde{\Pi}$  the projections onto these three components respectively. We assume  $\Pi_1 w = 0$  and the  $\Pi_1$  part of  $w$  is actually represented by  $\eta$ .

Now we can state Theorem 1.1 rigorously.

**Theorem 2.3.** *For any  $\tau_0 \in (0, 1/4)$  there is  $\varepsilon_0 > 0$  such that when  $0 < \varepsilon < \varepsilon_0$  and  $\varepsilon$  is proper size for  $\tau_0$  and  $L_\Gamma$ , we can choose at least two points  $p_1, p_2$  on the geodesic and  $w_{0,1}, \eta_1, \tilde{w}_1, w_{0,2}, \eta_2, \tilde{w}_2$  such that for  $i = 1, 2$   $w_{0,i}, \eta_i, \tilde{w}_i$  beyond to 0th part, 1st part and high part respectively and*

$$H(\mathcal{D}_{\phi_{\tau_0}, p_i, \varepsilon}(w_{0,i} + \tilde{w}_i, \eta_i)) = \frac{2}{\varepsilon}, i = 1, 2$$

and for uniform constant  $C$  we have

$$\begin{aligned} \|w_{0,i}\|_{C_\varepsilon^{2,\alpha}} &\leq C\varepsilon, \\ \|\eta_i\|_{C_{x_0, \varepsilon}^{2,\alpha}} &\leq C\varepsilon^2, \\ \|\tilde{w}_i\|_{C_\varepsilon^{2,\alpha}} &\leq C\varepsilon^2. \end{aligned}$$

Moreover  $\mathcal{D}_{\phi_{\tau_0}, p_1, \varepsilon}(w_{0,1} + \tilde{w}_1, \eta_1)$  is different from  $\mathcal{D}_{\phi_{\tau_0}, p_2, \varepsilon}(w_{0,2} + \tilde{w}_2, \eta_2)$ .

## 2.2 The mean curvature of $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)$

### 2.2.1 The first fundamental form of $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)$

Now we calculate the first fundamental form of  $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)$  with respect to the coordinate  $(s, \theta)$ . At the point

$$q = F(\varepsilon\psi(s), \varepsilon(\phi(s) + w(s, \theta))\Upsilon(\theta) + \eta(\varepsilon\psi(s))).$$

Suppose  $p = F(\varepsilon\psi(s), 0)$ . First we have

$$\begin{cases} \partial_s &= \varepsilon(\dot{\psi}X_0 + (\dot{\phi} + \frac{\partial w}{\partial s})\Upsilon + \dot{\psi}\frac{\partial \eta}{\partial x_0}) \\ \partial_\theta &= \varepsilon((\phi + w)\Upsilon_\theta + \frac{\partial w}{\partial \theta}\Upsilon), \end{cases} \quad (5)$$

and  $x_k(q) = \varepsilon(\phi(s) + w(s, \theta))\Upsilon^k + \eta^k, k = 1, 2$ .

From now on we use the symbol  $L(w, \eta)$  to represent a function which is linear in  $w$  or  $\eta$  and  $Q(w, \eta)$  to represent a function which is quadratic in  $w$  and  $\eta$ .  $L(w, \eta)$  and  $Q(w, \eta)$  contain derivatives of  $w$  up to second order and derivatives of  $\eta$  up to first order. If  $\partial_{x_0}^2 \eta$  appears, we will denote  $L(\partial_{x_0}^2 \eta), Q(\partial_{x_0}^2 \eta, w)$ .

From (3) we know

**Lemma 2.4.**

$$\begin{aligned}
\langle X_0, X_0 \rangle_q &= 1 + \varepsilon^2 \phi^2 R(\Upsilon, X_0, \Upsilon, X_0)_p + 2\varepsilon \phi R(\Upsilon, X_0, \eta, X_0)_p \\
&\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3), \\
\langle X_i, X_j \rangle_q &= \delta_{ij} + \frac{1}{3} \varepsilon^2 \phi^2 R(\Upsilon, X_i, \Upsilon, X_j)_p + \frac{1}{3} \varepsilon \phi (R(\Upsilon, X_i, \eta, X_j)_p \\
&\quad + R(\eta, X_i, \Upsilon, X_j)_p) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3), \\
\langle X_0, X_i \rangle_q &= \frac{2}{3} \varepsilon^2 \phi^2 R(\Upsilon, X_0, \Upsilon, X_i)_p + \frac{2}{3} \varepsilon \phi (R(\Upsilon, X_0, \eta, X_i)_p \\
&\quad + R(\eta, X_0, \Upsilon, X_i)_p) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3).
\end{aligned}$$

We use these expansions to obtain the expansions of the first fundamental form,

**Lemma 2.5.**

$$\begin{aligned}
\varepsilon^{-2} \langle \partial_s, \partial_s \rangle &= \phi^2 + \varepsilon^2 \phi^2 \dot{\psi}^2 R(\Upsilon, X_0, \Upsilon, X_0) + 2\varepsilon \phi \dot{\psi}^2 R(\Upsilon, X_0, \eta, X_0) \\
&\quad + \frac{4}{3} \varepsilon \phi \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \eta, \Upsilon) + 2\dot{\phi} \frac{\partial w}{\partial s} + 2\dot{\phi} \dot{\psi} \langle \Upsilon, \frac{\partial \eta}{\partial x_0} \rangle_e \\
&\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3), \\
\varepsilon^{-2} \langle \partial_s, \partial_\theta \rangle &= \frac{2}{3} \varepsilon^2 \phi^3 \dot{\psi} R(\Upsilon, X_0, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon \phi^2 \dot{\psi} (R(\Upsilon, X_0, \eta, \Upsilon_\theta) \\
&\quad + R(\eta, X_0, \Upsilon, \Upsilon_\theta)) + \frac{1}{3} \varepsilon \phi^2 \dot{\phi} R(\eta, \Upsilon, \Upsilon, \Upsilon_\theta) + \dot{\phi} \frac{\partial w}{\partial \theta} \\
&\quad + \dot{\phi} \dot{\psi} \langle \frac{\partial \eta}{\partial x_0}, \Upsilon_\theta \rangle_e + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3), \\
\varepsilon^{-2} \langle \partial_\theta, \partial_\theta \rangle &= \phi^2 + 2\phi w + \frac{1}{3} \varepsilon^2 \phi^4 R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon \phi^3 R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) \\
&\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3).
\end{aligned}$$

*Note that all the curvatures here and after are taken on  $p \in \Gamma$ .*

*Proof.* The proof is direct calculation, by using Lemma 2.4. For example for the first one  $\varepsilon^{-2} \langle \partial_s, \partial_s \rangle$ , first we have

$$\varepsilon^{-2} \langle \partial_s, \partial_s \rangle = \langle \dot{\psi} X_0 + (\dot{\phi} + \frac{\partial w}{\partial s}) \Upsilon + \dot{\psi} \frac{\partial \eta}{\partial x_0}, \dot{\psi} X_0 + (\dot{\phi} + \frac{\partial w}{\partial s}) \Upsilon + \dot{\psi} \frac{\partial \eta}{\partial x_0} \rangle.$$

And we get 6 different terms on the right hand side. For each one we can use Lemma 2.4. Finally we can get the results.  $\square$

### 2.2.2 Normal vector

To calculate the mean curvature, one need to know the normal vector. Now we are going to find the expansions of the unit normal vector of  $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)$ . First we take

$$N_0 = \frac{1}{\phi} (\dot{\phi} X_0 - \dot{\psi} \Upsilon), \tag{6}$$

which is the unit normal vector of  $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(0, 0)$  when curvature vanishes. We expect the unit normal vector  $N$  of  $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)$  is a small perturbation of  $N_0$ . Namely, we assume

$$N = \frac{1}{k}(N_0 + a_1 \partial_s + a_2 \partial_\theta) \quad (7)$$

where  $k$  is the norm of  $N_0 + a_1 \partial_s + a_2 \partial_\theta$ .

$$\begin{aligned} 0 &= \langle kN, \partial_s \rangle = \langle N_0, \partial_s \rangle + a_1 \langle \partial_s, \partial_s \rangle + a_2 \langle \partial_s, \partial_\theta \rangle, \\ 0 &= \langle kN, \partial_\theta \rangle = \langle N_0, \partial_\theta \rangle + a_1 \langle \partial_s, \partial_\theta \rangle + a_2 \langle \partial_\theta, \partial_\theta \rangle. \end{aligned} \quad (8)$$

**Lemma 2.6.**

$$\begin{aligned} \varepsilon^{-1} \langle N_0, \partial_s \rangle &= -\frac{\dot{\psi}}{\phi} \frac{\partial w}{\partial s} - \frac{\dot{\psi}^2}{\phi} \langle \frac{\partial \eta}{\partial x_0}, \Upsilon \rangle_e + \varepsilon^2 \phi \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \Upsilon, X_0) \\ &\quad + 2\varepsilon \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \eta, X_0) + \frac{2}{3} \varepsilon (\dot{\phi}^2 - \dot{\psi}^2) R(\Upsilon, X_0, \eta, \Upsilon) \\ &\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \\ \varepsilon^{-1} \langle N_0, \partial_\theta \rangle &= \frac{2}{3} \varepsilon^2 \phi^2 \dot{\phi} R(\Upsilon, X_0, \Upsilon, \Upsilon_\theta) \\ &\quad + \frac{2}{3} \varepsilon \phi \dot{\phi} (R(\Upsilon, X_0, \eta, \Upsilon_\theta) + R(\eta, X_0, \Upsilon, \Upsilon_\theta)) \\ &\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \end{aligned}$$

*Proof.* The proof is again direct calculations using (5) and (6). □

We denote  $g_{ss} = \langle \partial_s, \partial_s \rangle, g_{s\theta} = g_{\theta s} = \langle \partial_s, \partial_\theta \rangle, g_{\theta\theta} = \langle \partial_\theta, \partial_\theta \rangle$ . From Lemma 2.5 we have

$$\begin{pmatrix} g_{ss} & g_{s\theta} \\ g_{s\theta} & g_{\theta\theta} \end{pmatrix} = \varepsilon^2 \phi^2 \begin{pmatrix} 1 + \sigma_1 & \sigma_2 \\ \sigma_2 & 1 + \sigma_3 \end{pmatrix}.$$

where

$$\begin{cases} \sigma_1 &= \varepsilon^2 \dot{\psi}^2 R(\Upsilon, X_0, \Upsilon, X_0) + 2\varepsilon \phi^{-1} \dot{\psi}^2 R(\Upsilon, X_0, \eta, X_0) \\ &\quad + \frac{4}{3} \varepsilon \phi^{-1} \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \eta, \Upsilon) + 2\phi^{-2} \dot{\phi} \frac{\partial w}{\partial s} \\ &\quad + 2\phi^{-2} \dot{\phi} \dot{\psi} \langle \Upsilon, \frac{\partial \eta}{\partial x_0} \rangle_e + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \\ \sigma_2 &= \frac{2}{3} \varepsilon^2 \phi \dot{\psi} R(\Upsilon, X_0, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon \dot{\psi} (R(\Upsilon, X_0, \eta, \Upsilon_\theta) + R(\eta, X_0, \Upsilon, \Upsilon_\theta)) \\ &\quad + \frac{1}{3} \varepsilon \dot{\phi} R(\eta, \Upsilon, \Upsilon, \Upsilon_\theta) + \phi^{-2} \dot{\phi} \frac{\partial w}{\partial \theta} + \phi^{-1} \dot{\psi} \langle \frac{\partial \eta}{\partial x_0}, \Upsilon_\theta \rangle_e \\ &\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \\ \sigma_3 &= 2\phi^{-1} w + \frac{1}{3} \varepsilon^2 \phi^2 R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon \phi R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) \\ &\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \end{cases} \quad (9)$$

Notice that  $\sigma_1 \sigma_3 - \sigma_2^2 = \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)$ . We have

$$\det \begin{pmatrix} g_{ss} & g_{s\theta} \\ g_{s\theta} & g_{\theta\theta} \end{pmatrix} \cong \varepsilon^4 \phi^4 (1 + \sigma_1 + \sigma_3)$$

and

$$\det \begin{pmatrix} g_{ss} & g_{s\theta} \\ g_{s\theta} & g_{\theta\theta} \end{pmatrix}^{-1} \cong \varepsilon^{-4} \phi^{-4} (1 - \sigma_1 - \sigma_3).$$

So we get the inverse matrix

$$\begin{aligned} \begin{pmatrix} g^{ss} & g^{s\theta} \\ g^{s\theta} & g^{\theta\theta} \end{pmatrix} &= \det \begin{pmatrix} g_{ss} & g_{s\theta} \\ g_{s\theta} & g_{\theta\theta} \end{pmatrix}^{-1} \begin{pmatrix} g_{\theta\theta} & -g_{s\theta} \\ -g_{s\theta} & g_{ss} \end{pmatrix} \\ &\cong \varepsilon^{-2} \phi^{-2} \begin{pmatrix} 1 - \sigma_1 & -\sigma_2 \\ -\sigma_2 & 1 - \sigma_3 \end{pmatrix}. \end{aligned} \quad (10)$$

From this and Lemma 2.6 we can get

**Lemma 2.7.**

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \cong -\varepsilon^{-2} \phi^{-2} \begin{pmatrix} \langle N_0, \partial_s \rangle \\ \langle N_0, \partial_\theta \rangle \end{pmatrix} = \begin{pmatrix} O(\varepsilon) + \varepsilon^{-1} L(w, \eta) + \varepsilon^{-1} Q(w, \eta) \\ O(\varepsilon) + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) \end{pmatrix}$$

*Proof.* From (8), we have

$$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \varepsilon^{-2} \phi^{-2} \begin{pmatrix} 1 - \sigma_1 & -\sigma_2 \\ -\sigma_2 & 1 - \sigma_3 \end{pmatrix} \begin{pmatrix} -\langle N_0, \partial_s \rangle \\ -\langle N_0, \partial_\theta \rangle \end{pmatrix}. \quad (11)$$

By direct calculation, we have  $\sigma_i \langle N_0, \partial_s \rangle$  and  $\sigma_i \langle N_0, \partial_\theta \rangle$  are in fact  $\varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4)$ . So we can get the conclusion.  $\square$

Also we need to get the expansion of  $k$  which is the norm of  $N_0 + a_1 \partial_s + a_2 \partial_\theta$ .

$$\begin{aligned} k^2 &= \langle N_0 + a_1 \partial_s + a_2 \partial_\theta, N_0 + a_1 \partial_s + a_2 \partial_\theta \rangle \\ &= \langle N_0, N_0 \rangle + a_1^2 \langle \partial_s, \partial_s \rangle + a_2^2 \langle \partial_\theta, \partial_\theta \rangle + 2a_1 \langle N_0, \partial_s \rangle \\ &\quad + 2a_2 \langle N_0, \partial_\theta \rangle + 2a_1 a_2 \langle \partial_s, \partial_\theta \rangle. \end{aligned}$$

From (8) we can simplify this to

$$k^2 = \langle N_0, N_0 \rangle + a_1 \langle N_0, \partial_s \rangle + a_2 \langle N_0, \partial_\theta \rangle.$$

From

$$\begin{aligned} \langle N_0, N_0 \rangle &= \langle \frac{1}{\phi}(\dot{\phi} X_0 - \dot{\psi} \Upsilon), \frac{1}{\phi}(\dot{\phi} X_0 - \dot{\psi} \Upsilon) \rangle \\ &= \frac{\dot{\phi}^2}{\phi^2} \langle X_0, X_0 \rangle + \frac{\dot{\psi}^2}{\phi^2} \langle \Upsilon, \Upsilon \rangle - 2 \frac{\dot{\phi} \dot{\psi}}{\phi^2} \langle X_0, \Upsilon \rangle \\ &= 1 + \varepsilon^2 \dot{\phi}^2 R(\Upsilon, X_0, \Upsilon, X_0) + 2\varepsilon \frac{\dot{\phi}^2}{\phi} R(\Upsilon, X_0, \eta, X_0) \\ &\quad - \frac{4}{3} \varepsilon \frac{\dot{\phi} \dot{\psi}}{\phi} R(\Upsilon, X_0, \eta, \Upsilon) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3), \end{aligned}$$

$$\begin{aligned}
a_1 < N_0, \partial_s > &= (O(\varepsilon) + \varepsilon^{-1}L(w, \eta) + \varepsilon^{-1}Q(w, \eta))\varepsilon(-\frac{\dot{\psi}}{\phi}\frac{\partial w}{\partial s} - \frac{\dot{\psi}^2}{\phi} < \frac{\partial \eta}{\partial x^0}, \Upsilon >_e \\
&\quad + \varepsilon^2 \phi \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \Upsilon, X_0) + 2\varepsilon \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \eta, X_0) \\
&\quad + \frac{2}{3}(\dot{\phi}^2 - \dot{\psi}^2)R(\Upsilon, X_0, \eta, \Upsilon) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \\
&= \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^4),
\end{aligned}$$

$$\begin{aligned}
a_2 < N_0, \partial_\theta > &= (O(\varepsilon) + \varepsilon L(w, \eta) + \varepsilon^{-1}Q(w, \eta))\varepsilon(\frac{2}{3}\varepsilon^2 \dot{\phi}^2 \dot{\phi} R(\Upsilon, X_0, \Upsilon, \Upsilon_\theta) \\
&\quad + \frac{2}{3}\varepsilon \phi \dot{\phi} (R(\Upsilon, X_0, \eta, \Upsilon_\theta) + R(\eta, X_0, \Upsilon, \Upsilon_\theta)) \\
&\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \\
&= \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^4),
\end{aligned}$$

we have

$$\begin{aligned}
k^2 &= 1 + \varepsilon^2 \dot{\phi}^2 R(\Upsilon, X_0, \Upsilon, X_0) + 2\varepsilon \frac{\dot{\phi}^2}{\phi} R(\Upsilon, X_0, \eta, X_0) - \frac{4}{3}\varepsilon \frac{\dot{\phi} \dot{\psi}}{\phi} R(\Upsilon, X_0, \eta, \Upsilon) \\
&\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3).
\end{aligned}$$

So

$$\begin{aligned}
k &= 1 + \frac{\varepsilon^2}{2} \dot{\phi}^2 R(\Upsilon, X_0, \Upsilon, X_0) + \varepsilon \frac{\dot{\phi}^2}{\phi} R(\Upsilon, X_0, \eta, X_0) - \frac{2}{3}\varepsilon \frac{\dot{\phi} \dot{\psi}}{\phi} R(\Upsilon, X_0, \eta, \Upsilon) \\
&\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3). \tag{12}
\end{aligned}$$

### 2.2.3 The second fundamental form and the mean curvature of $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)$ .

Here we prove

$$\begin{aligned}
H(\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)) &= \frac{2}{\varepsilon} + \frac{1}{\varepsilon} \mathcal{L}_{S\mathcal{N}\Gamma} w + < \mathcal{J}\eta, \Upsilon > \\
&\quad + \varepsilon(F_1(\phi, \phi_\psi) \star R_1 + F_2(\phi, \phi_\psi) \star R_2) + E \\
&\quad + F_3(\phi, \phi_\psi) \star R_3(\eta) + T(w, \eta) \tag{13}
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{L}_{S\mathcal{N}\Gamma} w &= -\frac{\dot{\psi}}{\phi^3}(\frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial \theta^2}) - 2(\phi^2 - \tau_0)\frac{\dot{\phi}}{\phi^4}\frac{\partial w}{\partial s} - \frac{\dot{\psi}}{\phi^3}w, \\
< \mathcal{J}\eta, \Upsilon > &= -\frac{\dot{\psi}^3}{\phi^3} < \frac{\partial^2 \eta}{\partial x_0^2}, \Upsilon > - \frac{1}{\varepsilon}(\frac{\dot{\psi} \ddot{\psi}}{\phi^3} + 2(\phi^2 - \tau_0)\frac{\dot{\phi} \dot{\psi}}{\phi^4}) < \frac{\partial \eta}{\partial x_0}, \Upsilon > \\
&\quad + \phi^{-2}(2\dot{\phi} \ddot{\psi} + 2\frac{\dot{\phi}^2 \dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2\frac{\dot{\psi}^2}{\phi}(\phi^2 - \tau_0) - 2\phi \dot{\phi}^2)R(\Upsilon, X_0, \eta, X_0),
\end{aligned}$$

$$\begin{aligned}
F_1(\phi, \phi_\psi) \star R_1 &= \frac{1}{3} \dot{\psi} R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) \\
&\quad + \phi^{-2} (\phi \dot{\phi} \ddot{\psi} + 2 \dot{\phi}^2 \dot{\psi} + \dot{\psi}^3 - (\phi^2 - \tau_0) \dot{\psi}^2 - \phi^2 \dot{\phi}^2) R(\Upsilon, X_0, \Upsilon, X_0), \\
F_2(\phi, \phi_\psi) \star R_2 &= \frac{2}{3} \dot{\phi} R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta), \\
E &= O(\varepsilon^2), \\
F_3(\phi, \phi_\psi) \star R_3(\eta) &= \phi^{-2} \left( \frac{2}{3} \dot{\phi} \ddot{\phi} + \frac{2}{3} \frac{\dot{\phi}^3}{\phi} - \frac{2}{3} \dot{\psi} \ddot{\psi} - \frac{4}{3} (\phi^2 - \tau_0) \frac{\dot{\phi} \dot{\psi}}{\phi} - \frac{2}{3} \phi \dot{\phi} + \frac{4}{3} \phi \dot{\phi} \dot{\psi} \right) \\
&\quad R(\Upsilon, X_0, \eta, \Upsilon) + \frac{2}{3} \phi^{-1} \dot{\phi} R(\Upsilon_\theta, X_0, \eta, \Upsilon_\theta), \\
T(w, \eta) &= \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) + \varepsilon^2 L(\partial^2 \eta) + Q(w, \partial^2 \eta).
\end{aligned}$$

For our later needs, we assume

$$F_4(\phi, \phi_\psi) = \phi^{-2} \left( \frac{2}{3} \dot{\phi} \ddot{\phi} + \frac{2}{3} \frac{\dot{\phi}^3}{\phi} - \frac{2}{3} \dot{\psi} \ddot{\psi} - \frac{4}{3} (\phi^2 - \tau_0) \frac{\dot{\phi} \dot{\psi}}{\phi} + \frac{4}{3} \phi \dot{\phi} \dot{\psi} \right).$$

*Remark.* The reason why  $F_1, F_2, F_3, F_4$  only depends on  $\phi, \phi_\psi$  is

$$\begin{cases} \phi(s) = \phi(\psi) \\ \dot{\psi}(s) = \phi(\psi)^2 + \tau_0 \\ \dot{\phi}(s) = \frac{\partial \phi(\psi)}{\partial \psi} (\phi(\psi)^2 + \tau_0) \\ \ddot{\psi}(s) = \frac{\partial}{\partial s} (\phi(\psi)^2 + \tau_0) = 2\phi \frac{\partial \phi(\psi)}{\partial \psi} (\phi^2 + \tau_0) \\ \tau_0 = -\phi^2 + \frac{\phi}{\sqrt{1+\phi_\psi^2}} \end{cases}$$

*Remark.* Note that the terms in  $T(w, \eta)$  also depend on  $\phi$  and  $\phi_\psi$  and the curvatures. For example,  $Q(w, \eta)$  has Taylor expansion and each term of the expansion looks like

$$\varepsilon^{a_0} F_I(\phi, \phi_\psi) R_I w^{a_1} (\partial_s w)^{a_2} (\partial_s^2 w)^{a_3} (\partial_\theta w)^{a_4} (\partial_\theta^2 w)^{a_5} \eta^{a_6} (\partial_{x_0} \eta)^{a_7}$$

where  $I = (a_0, \dots, a_7) \geq 0$  and  $R_I$  represents the curvature term on the geodesic.  $L(w, \eta)$  and  $Q(w, \eta)$  do not involve  $\partial_{x_0}^2 \eta$ . By tracing each place where  $\partial_{x_0}^2 \eta$  appears, we conclude there is no term like  $F(\phi, \phi_\psi) R \cdot (\partial_{x_0}^2 \eta)^2$  in  $Q(w, \partial_{x_0}^2 \eta)$ . Also  $E = \sum_{a_0 \geq 2} \varepsilon^{a_0} F_{a_0}(\phi, \phi_\psi) R_{a_0}$ .

*Remark.* For the tail terms, we have

$$\begin{aligned}
&\Pi_0(\varepsilon(F_1(\phi, \phi_\psi) \star R_1 + F_2(\phi, \phi_\psi) \star R_2) + F_3(\phi, \phi_\psi) \star R_3(\eta)) \\
&= \varepsilon F_1(\phi, \phi_\psi) \star \Pi_0(R_1) + F_3(\phi, \phi_\psi) \star \Pi_0(R_3(\eta)), \\
&\Pi_1(\varepsilon(F_1(\phi, \phi_\psi) \star R_1 + F_2(\phi, \phi_\psi) \star R_2) + F_3(\phi, \phi_\psi) \star R_3(\eta)) \\
&= \varepsilon F_2(\phi, \phi_\psi) \star \Pi_1(R_2).
\end{aligned} \tag{14}$$

The proof of (13) is very long calculation. We write it in Appendix A.

### 3 Jacobi operator

In this section we study the linear operators which appear in the expression of  $H(\mathcal{D}_{\phi\tau_0, p_0, \varepsilon}(w, \eta))$ .

#### 3.1 Definitions

The two linear operators appearing in (13) are

$$\begin{aligned} w \mapsto \mathcal{L}_{SNT} w &:= -\frac{\dot{\psi}}{\phi^3} \left( \frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial \theta^2} \right) - 2(\phi^2 - \tau_0) \frac{\dot{\phi}}{\phi^4} \frac{\partial w}{\partial s} - \frac{\dot{\psi}}{\phi^3} w, \\ \eta \mapsto \mathcal{J} \eta &:= -\frac{\dot{\psi}^3}{\phi^3} \frac{\partial^2 \eta}{\partial x_0^2} - \frac{1}{\varepsilon} \left( \frac{\dot{\psi} \ddot{\psi}}{\phi^3} + 2(\phi^2 - \tau_0) \frac{\dot{\phi} \dot{\psi}}{\phi^4} \right) \frac{\partial \eta}{\partial x_0} \\ &\quad - \phi^{-2} (2\dot{\phi} \ddot{\psi} + 2 \frac{\dot{\phi}^2 \dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2 \frac{\dot{\psi}^2}{\phi} (\phi^2 - \tau_0) - 2\phi \dot{\phi}^2) R(\eta, X_0) X_0. \end{aligned}$$

The operator  $\mathcal{L}_{SNT}$  is conjugate to the Jacobi operator which corresponds to the second variation of the energy functional. When  $\tau_0 = \frac{1}{4}$ , i.e. the Delaunay surface is a cylinder, the second operator  $\mathcal{J}$  is reduced to the Jacobi operator  $\mathcal{J}_A$  of the geodesic, which is invertible as the geodesic is non-degenerate.

It is easy to see that

$$\mathcal{L}_{SNT} : C_\varepsilon^{2,\alpha}(SNT) \mapsto C_\varepsilon^{0,\alpha}(SNT)$$

is bounded uniformly in  $\varepsilon$ .

$$\mathcal{J} : C_{x_0, \varepsilon}^{2,\alpha}(N\Gamma) \mapsto C_\varepsilon^\alpha(N\Gamma)$$

and

$$\|\mathcal{J}(\eta)\|_{C_\varepsilon^\alpha} \leq \frac{C}{\varepsilon} \|\eta\|_{C_{x_0, \varepsilon}^{2,\alpha}}.$$

We let

$$\mathcal{L}_0 = \mathcal{L}_{SNT}|_{\Pi_0(C_\varepsilon^{2,\alpha}(SNT))}, \tilde{\mathcal{L}} = \mathcal{L}_{SNT}|_{\tilde{\Pi}(C_\varepsilon^{2,\alpha}(SNT))}.$$

We are going to study the mapping properties of  $\tilde{\mathcal{L}}, \mathcal{J}, \mathcal{L}_0$  in three different modes, i.e. high mode, 1st mode, 0th mode.

#### 3.2 High mode

In this mode, we are going to prove that

$$\tilde{\mathcal{L}} : \tilde{\Pi} C_\varepsilon^{2,\alpha}(SNT) \rightarrow \tilde{\Pi} C_\varepsilon^{0,\alpha}(SNT)$$

is an isomorphism whose inverse is bounded independent of  $\varepsilon$ .

First it is clear that

$$\tilde{\mathcal{L}}(\tilde{\Pi} C_\varepsilon^{2,\alpha}(SNT)) \subseteq \tilde{\Pi} C_\varepsilon^{0,\alpha}(SNT).$$

From  $\frac{\partial}{\partial s} = \varepsilon \dot{\psi} \frac{\partial}{\partial x_0} = \dot{\psi} \frac{\partial}{\partial \psi}$ , we have, for  $w, v \in \tilde{W}_\varepsilon^{1,2}(S\Gamma)$

$$\begin{aligned}\tilde{\mathcal{L}}w &= -\frac{\dot{\psi}}{\phi^3} \left( \frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial \theta^2} \right) - 2(\phi^2 - \tau_0) \frac{\dot{\phi}}{\phi^4} \frac{\partial w}{\partial s} - \frac{\dot{\psi}}{\phi^3} w \\ &= -\frac{\phi}{\dot{\psi}^3} \left( \frac{\dot{\psi}^3}{\phi^2} \frac{\partial}{\partial \psi} \left( \frac{\dot{\psi}^3}{\phi^2} \frac{\partial}{\partial \psi} w \right) + \frac{\dot{\psi}^4}{\phi^4} w + \frac{\dot{\psi}^4}{\phi^4} \frac{\partial^2 w}{\partial \theta^2} \right)\end{aligned}$$

Consider the bounded bilinear functional

$$B(v, w) = \int_{S\Gamma} (v(\phi \tilde{\mathcal{L}})w) d\theta d\psi.$$

We have, for some positive constant  $C(\tau_0)$  which only depends on  $\tau_0$ ,

$$\begin{aligned}B(w, w) &= \int_{S\Gamma} \left( \frac{\dot{\psi}^3}{\phi^2} \left| \frac{\partial}{\partial \psi} w \right|^2 - \frac{\dot{\psi}}{\phi^2} w^2 + \frac{\dot{\psi}}{\phi^2} \left| \frac{\partial w}{\partial \theta} \right|^2 \right) d\theta d\psi \\ &\geq C(\tau_0) \int_{S\Gamma} \left( \left| \frac{\partial w}{\partial \psi} \right|^2 + \left| \frac{\partial w}{\partial \theta} \right|^2 + |w|^2 \right) d\theta d\psi.\end{aligned}\tag{15}$$

The inequality holds because for  $w \in \tilde{W}_\varepsilon^{1,2}(S\Gamma)$ , we have

$$\int_{S\Gamma \cap \{\psi=\psi_0\}} \left| \frac{\partial w}{\partial \theta} \right|^2 d\theta \geq 4 \int_{S\Gamma \cap \{\psi=\psi_0\}} |w|^2 d\theta,$$

for every  $\psi_0$ .

From (15) and the Lax-Milgram theorem we know  $\phi \tilde{\mathcal{L}}$  is invertible and

$$\|w\|_{W_\varepsilon^{1,2}} \leq C(\tau_0) \|\phi \tilde{\mathcal{L}}w\|_{W_\varepsilon^{-1,2}}.$$

And from standard regularity theory of elliptic PDE we can get

$$\|w\|_{C_\varepsilon^{2,\alpha}} \leq C(\tau_0) \|\phi \tilde{\mathcal{L}}w\|_{C_\varepsilon^{0,\alpha}} \leq C(\tau_0) \|\tilde{\mathcal{L}}w\|_{C_\varepsilon^{0,\alpha}}.$$

### 3.3 1st mode

In this mode, we are going to prove that

$$\mathcal{J}\eta : C_{x_0, \varepsilon}^{2,\alpha}(N\Gamma) \rightarrow C_\varepsilon^\alpha(N\Gamma)$$

is invertible and the inverse is independent of  $\varepsilon$ . First we need several technical lemmas.

#### 3.3.1 Technical lemmas

First we need to find the relationship between the operator  $\mathcal{J}$  and the Jacobi operator of the geodesic  $\mathcal{J}_A$ . Notice that

$$\begin{aligned}\frac{\dot{\psi}^3}{\phi} \mathcal{J}\eta &= -\frac{\dot{\psi}^3}{\phi^2} \frac{\partial}{\partial x_0} \left( \frac{\dot{\psi}^3}{\phi^2} \frac{\partial \eta}{\partial x_0} \right) \Big|_{x_0} \\ &\quad - \frac{\dot{\psi}^3}{\phi^3} \left( 2\phi \ddot{\psi} + 2\frac{\dot{\phi}^2 \dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2\frac{\dot{\psi}^2}{\phi} (\phi^2 - \tau_0) - 2\phi \dot{\phi}^2 \right) R(\eta, X_0) X_0 \Big|_{x_0}.\end{aligned}$$



Let  $\frac{\dot{\psi}^3}{\phi^2} \frac{\partial}{\partial x_0} = \frac{\partial}{\partial y_0}$ , then  $dy_0 = \frac{\phi^2}{\dot{\psi}^3} dx_0$ . We can solve

$$\begin{cases} dy_0 &= \frac{\phi^2}{\dot{\psi}^3} dx_0, \\ y_0(0) &= 0, \end{cases} \quad (16)$$

and get  $y_0(x_0)$ . We know  $x_0 = \varepsilon\psi$ . So the period of  $\phi$  and  $\psi$  or the derivatives of them (in  $x_0$  coordinate) have period of order  $\varepsilon$ . So coefficients such as  $\frac{\phi^2}{\dot{\psi}^3}$  and  $\frac{\dot{\psi}^3}{\phi^3}(2\dot{\phi}\ddot{\psi} + 2\frac{\dot{\phi}^2\dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2\frac{\dot{\psi}^2}{\phi}(\phi^2 - \tau_0) - 2\phi\dot{\phi}^2)$  are highly oscillating in  $x_0$  coordinate. To understand the mean value of the coefficients in the right way is the key to understand the operator  $\mathcal{J}$ .

Suppose  $\psi \in [a_1, b_1]$  is one period of  $\phi$ . Suppose

$$\frac{\int_{a_1}^{b_1} \frac{\phi^2}{\dot{\psi}^3} d\psi}{\int_{a_1}^{b_1} d\psi} = I_1.$$

$I_1$  is approximately the ratio of the length of  $y_0$  and that of  $x_0$ . In some sense  $dy_0 \cong I_1 dx_0$ . We have

$$\frac{\dot{\psi}^3}{\phi} \mathcal{J}\eta = -\frac{\partial^2 \eta}{\partial y_0^2} \Big|_{y_0(x_0)} - \Psi_1(\phi, \phi_\psi) R(\eta, X_0) X_0 \Big|_{x_0},$$

where

$$\Psi_1(\phi, \phi_\psi) = \frac{\dot{\psi}^3}{\phi^3} (2\dot{\phi}\ddot{\psi} + 2\frac{\dot{\phi}^2\dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2\frac{\dot{\psi}^2}{\phi}(\phi^2 - \tau_0) - 2\phi\dot{\phi}^2).$$

Now we need the average of  $\Psi_1(\phi, \phi_\psi)$  in the coordinate  $y_0$ . Note that  $dy_0 = \frac{\phi^2}{\dot{\psi}^3} dx_0 = \varepsilon \frac{\phi^2}{\dot{\psi}^3} d\psi$ . If we assume  $y_0(a_1) = y_1, y_0(b_1) = y_2$ , then we have

$$\begin{aligned} & \frac{\int_{y_1}^{y_2} \Psi_1(\phi, \phi_\psi) dy_0}{\int_{y_1}^{y_2} dy_0} \\ &= \frac{\int_{a_1}^{b_1} \frac{1}{\phi} (2\dot{\phi}\ddot{\psi} + 2\frac{\dot{\phi}^2\dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2\frac{\dot{\psi}^2}{\phi}(\phi^2 - \tau_0) - 2\phi\dot{\phi}^2) d\psi}{\int_{a_1}^{b_1} \frac{\phi^2}{\dot{\psi}^3} d\psi} \\ &= I_2. \end{aligned} \quad (17)$$

This indicates that in some sense

$$\begin{aligned} & -\frac{\partial^2 \eta}{\partial y_0^2} - \Psi_1(\phi, \phi_\psi) R(\eta, X_0) X_0 \\ & \cong -\frac{\partial^2 \eta}{\partial y_0^2} \Big|_{y_0} - I_2 R(\eta, X_0) X_0 \Big|_{x_0}. \end{aligned}$$

But  $\frac{\partial}{\partial y_0} \cong I_1^{-1} \frac{\partial}{\partial x_0}$ . So

$$-\frac{\partial^2 \eta}{\partial y_0^2} \Big|_{y_0} - I_2 R(\eta, X_0) X_0 \Big|_{x_0} \cong -I_1^{-2} \left( \frac{\partial^2 \eta}{\partial x_0^2} \Big|_{x_0} + I_1^2 I_2 R(\eta, X_0) X_0 \right) \Big|_{x_0}.$$

So if

$$I_1^2 I_2 = 1,$$

we will have the chance to unearth the Jacobi operator of the geodesic  $\mathcal{J}_A$ . Fortunately, it is true. Due to easy calculation, it is equivalent to the following lemma

**Lemma 3.1.** (*“average 1” lemma*)

$$\begin{aligned} & \int_{a_1}^{b_1} \frac{1}{\phi} (2\dot{\phi}\ddot{\psi} + 2\frac{\dot{\phi}^2\dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2\frac{\dot{\psi}^2}{\phi}(\phi^2 - \tau_0) - 2\phi\dot{\phi}^2) d\psi \cdot \int_{a_1}^{b_1} \frac{\phi^2}{\psi^3} d\psi \\ &= (b_1 - a_1)^2. \end{aligned}$$

The proof of this lemma is direct calculations. It depends on some special properties of Delaunay surfaces. We write it in the Appendix B.

Let  $\tilde{\mathcal{J}}_A = -\frac{\partial^2}{\partial y_0^2} - I_2 R(\cdot, X_0) X_0|_{I_1^{-1}y_0(x_0)}$ .  $I_1^2 \tilde{\mathcal{J}}_A$  has the same form as  $\mathcal{J}_A$  (It is not equal to  $\mathcal{J}_A$  because they are operators in different coordinates!). So  $\tilde{\mathcal{J}}_A$  is invertible. Suppose  $G(y_0, z_0)$  is the Green function of the operator  $\tilde{\mathcal{J}}_A$  ( $\tilde{\mathcal{J}}_A$  acts on  $z_0$ ), i.e.

$$\tilde{\mathcal{J}}_A G(y_0, z_0) = \delta^I(z_0 - y_0).$$

*Remark.*  $G(y_0, z_0)$  is a section of  $(N\Gamma)^* \otimes N\Gamma$ . Any section of  $(N\Gamma)^* \otimes N\Gamma$  can be regarded as a section of  $\text{Hom}(N\Gamma, N\Gamma)$ . If we denote the identity of  $\text{Hom}(N\Gamma, N\Gamma)$  by  $I$ , then  $\delta^I(y_0)$  denotes  $\delta(y_0)I$ , where  $\delta(y_0)$  is the Dirac function supported on  $y_0$ . From Section 2.1.2, there is a notion of connection of the normal bundle of the geodesic. So one can take derivatives of the sections of  $N\Gamma$  and the sections of  $\text{Hom}(N\Gamma, N\Gamma)$  with respect to  $x_0$ .

First we have the following estimates for  $G(y_0, z_0)$ ,

**Lemma 3.2.**  $G(y_0, z_0)$  has the following properties,

1. For fixed  $y_0$ ,  $G(y_0, z_0), G_{y_0}(y_0, z_0) \in C_{z_0}^\infty((N\Gamma)^* \otimes N\Gamma)$  away from  $z_0 = y_0$ ;
2. For fixed  $y_0$ ,  $G(y_0, z_0) \in C_{z_0}^{0,1}((N\Gamma)^* \otimes N\Gamma)$ ,  $\|G(y_0, z_0)\|_{C_{z_0}^{0,1}} \leq C$ .  $C$  does not depend on  $y_0$ ;
3.  $G_{y_0}(y_0, z_0), G_{y_0 z_0}(y_0, z_0), z_0 \neq y_0$  are bounded independent of  $y_0$ .  $G_{y_0}(y_0, z_0)$  is discontinuous at  $z_0 = y_0$ .  $G_{y_0 z_0}(y_0, z_0)$  has both left and right limits on  $z_0 = y_0$ .

*Proof.* We have

$$\begin{aligned} \tilde{\mathcal{J}}_A G(y_0, z_0) &= \delta^I(z_0 - y_0), \\ \tilde{\mathcal{J}}_A G_{y_0}(y_0, z_0) &= -(\delta^I)'(z_0 - y_0). \end{aligned}$$

Note that  $\delta^I(z_0 - y_0) \in W_{z_0}^{-\frac{1}{2}, q_0}$  and  $(\delta^I)'(z_0 - y_0) \in W_{z_0}^{-\frac{3}{2}, q_0}$  for fixed  $1 < q_0 < 2$ . So there are  $\alpha$  and  $q'$  with  $0 < \alpha < 1$  and  $1 < q' < +\infty$ , such that

$$\begin{aligned} \|G(y_0, z_0)\|_{C_{z_0}^\alpha} &\leq C \|G(y_0, z_0)\|_{W_{z_0}^{\frac{3}{2}, q_0}} \leq C \|\delta^I\|_{W_{z_0}^{-\frac{1}{2}, q_0}} \leq C(q_0), \\ \|G_{y_0}(y_0, z_0)\|_{L_{z_0}^{q'}} &\leq C \|G_{y_0}(y_0, z_0)\|_{W_{z_0}^{\frac{1}{2}, q_0}} \leq C \|(\delta^I)'\|_{W_{z_0}^{-\frac{3}{2}, q_0}} \leq C(q_0). \end{aligned}$$

Cutting the geodesic at point  $y_0$ , we can regard it as an interval  $[0, L_\Gamma]$ . We have

$$\begin{aligned}\|G(y_0, z_0)\|_{C_{z_0}^2([0, L_\Gamma])} &\leq C\|G(y_0, z_0)\|_{C^0(\Gamma)} \leq C(q_0), \\ \|G_{y_0}\|_{C_{z_0}^{1,\alpha}([0, L_\Gamma])} &\leq C\|G_{y_0}\|_{L_{z_0}^{q'}([0, L_\Gamma])} \leq C(q_0)\end{aligned}$$

The first one implies  $G_{z_0}(y_0, y_0 - 0), G_{z_0}(y_0, y_0 + 0)$  exist and are bounded. So

$$G(y_0, z_0) \in C_{z_0}^{0,1}((N\Gamma)^* \otimes N\Gamma).$$

The second one implies  $G_{y_0}, G_{y_0, z_0}, z_0 \neq y_0$  are bounded and  $G_{y_0, z_0}$  has both left and right limits on  $z_0 = y_0$ .  $G_{y_0}(y_0, z_0)$  is discontinuous at  $z_0 = y_0$  because the right hand side of the equation for  $G_{y_0}$  is  $-(\delta^I)'(z_0 - y_0)$ .

It is very easy to deduce the first item of this lemma from the regularity of the solution. □

**Lemma 3.3.** *Suppose that  $F(\phi, \phi_\psi)$  is a smooth function of  $\phi$  and  $\phi_\psi$  and it has the cancellation property in  $y_0$  coordinate that*

$$\int_{y_1}^{y_2} F(\phi, \phi_\psi) dy_0 = 0$$

where  $[y_1, y_2]$  is one period of  $\phi(y_0)$ . Let  $R$  be a function in  $C_{x_0}^1$ . Then

$$h(y_0) = \int_{\Gamma} G(y_0, z_0) R(z_0) F(\phi, \phi_\psi)(z_0) dz_0$$

satisfies

$$\|h(y_0)\|_{C_{y_0}^1} \leq C\varepsilon \|R\|_{C_{y_0}^1} \leq C\varepsilon \|R\|_{C_{x_0}^1},$$

where  $C$  depends on  $\tau_0$ .

*Proof.* Suppose  $\chi(y_0)$  is the primitive function of  $F(\phi, \phi_\psi)$  in  $y_0$  coordinate. Because  $F(\phi, \phi_\psi)$  has cancellation property,  $\chi(y_0)$  is a global periodic function on  $\Gamma$ . We may assume  $\chi(y_0)$  is a special one such that it also has cancellation property. It is easy verified that

$$\|\chi(y_0)\|_{C^0} \leq C(\tau_0)\varepsilon. \quad (18)$$

We have

$$h(y_0) = - \int_{\Gamma} \chi(z_0) (G_{z_0} R + G R_{z_0}) dz_0,$$

and

$$\begin{aligned}h'(y_0) &= \int_{\Gamma} G_{y_0} R(z_0) F(\phi, \phi_\psi)(z_0) dz_0 \\ &= \int_0^{L_\Gamma} G_{y_0}(y_0, y_0 + z) R(y_0 + z) F(\phi, \phi_\psi)(y_0 + z) dz \\ &= G_{y_0}(y_0, y_0 + z) R(y_0 + z) \chi(y_0 + z) \Big|_0^{L_\Gamma} - \int_0^{L_\Gamma} \chi(y_0 + z) (G_{y_0 z} R + G_{y_0} R_z) dz.\end{aligned}$$

From  $|R_{z_0}| \leq C|R_{x_0}| \leq C$ , (18) and Lemma 3.2, we can prove this lemma. □

### 3.3.2 The main theorem of 1st mode

Now we can discuss the operator  $\mathcal{J}$ . First it is clear that, there is a map  $P_1 : L^2(SN\Gamma) \rightarrow L^2(N\Gamma)$  such that for any  $f \in L^2(SN\Gamma)$

$$\Pi_1(f) = \langle P_1(f), \Upsilon \rangle.$$

We have

**Theorem 3.4.** *There is  $\delta > 0$  such that when  $0 < \varepsilon < \delta$ , we have*

1. *For each  $f \in C_\varepsilon^\alpha(N\Gamma)$ , there exists a unique  $\eta \in C_{x_0, \varepsilon}^{2, \alpha}(N\Gamma)$  such that*

$$\mathcal{J}\eta = f$$

*and for some uniform  $C$  which does not depend on  $\varepsilon$*

$$\|\eta\|_{C_{x_0, \varepsilon}^{2, \alpha}} \leq C\|f\|_{C_\varepsilon^\alpha}.$$

2. *The equation*

$$\mathcal{J}\eta = -\frac{2}{3}\varepsilon\dot{\phi}P_1(R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta))$$

*has a unique solution  $\eta$  such that for some uniform  $C$*

$$\|\eta\|_{C_{x_0, \varepsilon}^{2, \alpha}} \leq C\varepsilon^2.$$

*Proof.* We prove the first item. Consider

$$\begin{aligned} & \frac{\dot{\psi}^3}{\phi} \mathcal{J}\eta \\ &= -\frac{\partial^2 \eta}{\partial y_0^2}|_{y_0(x_0)} - \Psi_1(\phi, \phi_\psi)R(\eta, X_0)X_0|_{x_0} \\ &= \tilde{\mathcal{J}}_A\eta + (I_2 - \Psi_1(\phi, \phi_\psi)|_{x_0})R|_{I_1^{-1}y_0(x_0)}(\eta(x_0), X_0)X_0 \\ & \quad + \Psi_1(\phi, \phi_\psi)|_{x_0}(R|_{I_1^{-1}y_0(x_0)} - R|_{x_0})(\eta(x_0), X_0)X_0 \\ &= \tilde{\mathcal{J}}_A\eta + (I_2 - \Psi_1(\phi, \phi_\psi)|_{x_0})R|_{I_1^{-1}y_0(x_0)}(\eta(x_0), X_0)X_0 + O(\varepsilon)L(\eta) \\ &= \frac{\dot{\psi}^3}{\phi} f. \end{aligned} \tag{19}$$

First we solve

$$\tilde{\mathcal{J}}_A\eta_1 = \frac{\dot{\psi}^3}{\phi} f.$$

From the invertibility of  $\tilde{\mathcal{J}}_A$  and Lemma 3.2 we know there is one unique solution  $\eta_1$

$$\begin{aligned} \|\eta_1\|_{C_{y_0}^2} &\leq \left\| \int_\Gamma G(y_0, z_0) \frac{\dot{\psi}^3}{\phi} f dz_0 \right\|_{C^0} \\ &\leq \|f\|_{C^0}. \end{aligned}$$

Then for each  $i \geq 1$  we solve

$$\begin{aligned} & \tilde{\mathcal{J}}_A(\eta_{i+1} - \eta_i) \\ &= (\Psi_1(\phi, \phi_\psi)|_{x_0} - I_2)R|_{I_1^{-1}y_0(x_0)}(\eta_i - \eta_{i-1}, X_0)X_0 + O(\varepsilon)L(\eta_i - \eta_{i-1}) \end{aligned}$$

where  $\eta_0 = 0$ . From (17) we know  $\Psi_1(\phi, \phi_\psi)|_{x_0} - I_2$  has 0 average in one period in  $y_0$  coordinate. From Lemma 3.3, we know

$$\|\eta_{i+1} - \eta_i\|_{C_{y_0}^1} \leq C\varepsilon\|\eta_i - \eta_{i-1}\|_{C_{y_0}^1}$$

where  $C$  depends on  $\tau_0$  and the norm of the curvature. So we can choose  $\delta$  such that  $0 < C\delta < \frac{1}{2}$  and get that  $\eta_i \rightarrow \eta$  in  $C_{y_0}^1$ . From

$$\|\eta_{i+1} - \eta_i\|_{C_{y_0}^2} \leq C\|\eta_i - \eta_{i-1}\|_{C_{y_0}^1},$$

we know  $\partial_{y_0}^2 \eta_i \rightarrow \partial_{y_0}^2 \eta$  is  $C^0$  sense. So we get a solution for  $\mathcal{J}\eta = f$  and we have the estimate

$$\|\eta\|_{C_{y_0}^1} \leq \|f\|_{C^0}.$$

So we have

$$\|\eta\|_{C_{x_0}^1} \leq \|f\|_{C^0}$$

and hence

$$\|\eta\|_{C_{x_0, \varepsilon}^{2, \alpha}} \leq \|f\|_{C_\varepsilon^\alpha}.$$

Now we prove the second item. In (19), we let  $f = -\frac{2}{3}\varepsilon\dot{\phi}P_1(R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta))$ . It is obvious that  $\frac{\psi^3}{\phi}\dot{\phi}$  has 0 average in one period. Apply Lemma 3.3 again, we can draw the conclusion.  $\square$

### 3.4 0th mode

#### 3.4.1 The main difficulties

The main difficulties lie in 0th mode. The 0th mode may be expressed as

$$\begin{aligned} \mathcal{L}_0 w &= -\frac{\dot{\psi}}{\phi^3} \frac{\partial^2 w}{\partial s^2} - 2(\phi^2 - \tau_0) \frac{\dot{\phi}}{\phi^4} \frac{\partial w}{\partial s} - \frac{\dot{\psi}}{\phi^3} w \\ &= -\varepsilon \Pi_0(\varepsilon(F_1(\phi, \phi_\psi) \star R_1 + F_2(\phi, \phi_\psi) \star R_2) + E \\ &\quad + F_3(\phi, \phi_\psi) \star R_3(\eta) + T(w, \eta)) \\ &= -\varepsilon^2 F_1(\phi, \phi_\psi) \star \Pi_0(R_1) - \varepsilon \Pi_0(E) - \varepsilon F_3(\phi, \phi_\psi) \star \Pi_0(R_3(\eta)) - \varepsilon \Pi_0(T(w, \eta)). \end{aligned}$$

where  $w = w(s)$  is a function on the geodesic. There are three difficulties.

1. The operator  $\mathcal{L}_0$  has kernel  $\frac{\dot{\phi}}{\psi} = \phi_\psi$  which corresponds to the translation of the surface along the geodesic. So one may only solve it up to the kernel.

2. Even when one tries to solve this ODE up to the kernel  $\phi_\psi$ , it is still not trivial. After analyzing  $\mathcal{L}_0$ , we find that the two fundamental solutions of  $\mathcal{L}_0 w = 0$  are  $h_1(\tau_0)\phi_\psi$  and  $h_2(\tau_0)\psi\phi_\psi + v$  where  $v$  is a periodic function. The first one is periodic and the second one has linear growth. So the  $C^0$  norm of the solution will become as big as  $\frac{1}{\varepsilon^2}\mathcal{L}_0 w$ . So one can only expect the solution to be bounded by a constant which does not depend on  $\varepsilon$  as the biggest term on the right is of  $O(\varepsilon^2)$  size. However, then, the first term in  $\varepsilon T(w, \eta)$ ,  $\varepsilon^2 L(w)$  is again as big as  $O(\varepsilon^2)$  and the second term  $Q(w, \eta)$  is as big as  $O(1)$ . The iteration breaks down.
3. The term  $F_3(\phi, \phi_\psi) \star R_3(\eta)$  also causes problem. Suppose we have some method to overcome the second difficulty and get a solution of 0th mode  $w = O(\varepsilon)$  when we assume  $\eta = 0$ . (Actually this is the case.) Now if  $\eta$  has a variation from 0 to  $O(\varepsilon^2)$ , the solution of 0th mode will have a variation of size  $\varepsilon$ . Then  $\Pi_1(T(w, \eta))$  will have a variation of size  $\varepsilon^2$  because of the term  $\frac{1}{\varepsilon}Q(w, \eta)$ . Then through 1st mode,  $\eta$  will again have a variation of size  $\varepsilon^2$ . So the iteration breaks down.

To overcome the three difficulties, we consider both a non linear ODE and its linearizations.

Choose a point  $p_0 \in \Gamma$  where  $\psi(p_0) = 0$ . Consider the mean curvature type non linear ODE

$$\begin{cases} \phi_{\psi\psi} & -\phi^{-1}(1 + \phi_\psi^2) + (2 + \rho)(1 + \phi_\psi^2)^{\frac{3}{2}} = 0, \\ \phi_\psi(0) & = \phi_\psi(\frac{L_\Gamma}{\varepsilon}) = 0, \\ \phi(0) & = \phi(\frac{L_\Gamma}{\varepsilon}) = \frac{1 - \sqrt{1 - 4\tau(0)}}{2}, \\ \rho & = -\varepsilon^2 F_1(\phi, \phi_\psi) \star \Pi_0(R_1) + \varepsilon F_4(\phi, \phi_\psi)\xi(x_0) \\ & \quad + \varepsilon^3 \mu(\psi) + \varepsilon^3 \omega \phi^{-1} \phi_\psi, \end{cases} \quad (20)$$

where  $\tau(0) = \tau(\phi(0), 0)$  and

$$\|\xi\|_{C_{x_0}^1} \leq C_1 \varepsilon^2, \|\mu\|_{C_\varepsilon^\alpha} \leq C_2, |\omega| \leq C_3, |\phi(0) - \frac{1 - \sqrt{1 - 4\tau_0}}{2}| \leq C(\tau_0)\varepsilon, \quad (21)$$

where  $C(\tau_0)$  is a small number which only depends on  $\tau_0$ .

From now on, we use  $\phi = \phi_{\xi, \mu, \omega, \tau(0)}$  to represent the solution to (20) and  $\phi_\tau$  to represent the defining function of standard Delaunay surface with constant parameter  $\tau$ .

Note that in  $\rho$  we prescribe two functions  $\xi(x_0)$ ,  $\mu(\psi)$  and two constants  $\omega$  and  $\tau(0)$ .

First we explain the reasons for  $\omega \phi^{-1} \phi_\psi$  and  $\tau(0)$ . To solve the ODE system above, we expect to have a global smooth function  $\phi$  on the closed geodesic. This means

$$\begin{cases} \phi(0) & = \phi(\frac{L_\Gamma}{\varepsilon}), \\ \phi_\psi(0) & = \phi_\psi(\frac{L_\Gamma}{\varepsilon}) = 0. \end{cases} \quad (22)$$

We can adjust  $\omega$  and  $\tau(0)$  (or equivalently  $\phi(0)$ ) to achieve this. From the linearization point of view,  $\omega\phi^{-1}\phi_\psi$  exists because  $\mathcal{L}_0$  has kernel. The first difficulty could not be solved at present.

The two functions  $\mu$  and  $\xi$  are used to do fixed point argument. The aim of 0th mode is to use  $\mu$  and  $\xi$  to cancel the 0th projection of tail terms. Recall (14),

$$\begin{aligned} & \Pi_0(\varepsilon(F_1(\phi, \phi_\psi) \star R_1 + F_2(\phi, \phi_\psi) \star R_2) + F_3(\phi, \phi_\psi) \star R_3(\eta) + E + T(w, \eta)) \\ &= \varepsilon F_1(\phi, \phi_\psi) \star \Pi_0(R_1) + F_3(\phi, \phi_\psi) \star \Pi_0(R_3(\eta)) + \Pi_0(E) + \Pi_0(T(w, \eta)). \end{aligned}$$

$\varepsilon F_1(\phi, \phi_\psi) \star \Pi_0(R_1)$  is a big term, however, we can solve it in the ODE directly, which is an advantage of the consideration of non-linear ODE. As will be seen in Step 3 later, because of certain cancellation property, the influence of  $\varepsilon F_1(\phi, \phi_\psi) \star \Pi_0(R_1)$  is only as big as  $O(\varepsilon^2)$ . Now in  $T(w, \eta)$ , we need only consider  $w = \tilde{\Pi}(w)$ . As will be seen later, in this way, the second difficulty is overcome.  $\mu$  is used to deal with  $\Pi_0(E) + \Pi_0(T(w, \eta))$ .

Now we notice

$$F_3(\phi, \phi_\psi) \star \Pi_0(R_3(\eta)) = F_4(\phi, \phi_\psi) \Pi_0(R(\Upsilon, X_0, \eta, \Upsilon))$$

because of the following observation,

$$\Pi_0(R(\Upsilon, X_0, \eta, \Upsilon) - R(\Upsilon_\theta, X_0, \eta, \Upsilon_\theta)) = 0.$$

Then we look at the equation of Delaunay parameter  $\tau$ ,

$$\frac{d\tau}{d\psi} = -\phi\phi_\psi F_4(\phi, \phi_\psi) \Pi_0(R(\Upsilon, X_0, \eta, \Upsilon)).$$

We notice that  $R(\Upsilon, X_0, \eta, \Upsilon)$  is a good function whose derivative with respect to  $\psi$  is of  $\varepsilon$  order. One surprising fact is the following ‘‘average 0’’ lemma.

**Lemma 3.5.** (*‘‘average 0’’ lemma*) Suppose  $\phi_\tau$  is the defining function for standard Delaunay surfaces of parameter  $\tau$  and  $\psi \in [a_\tau, b_\tau]$  is one period of  $\phi_\tau(\psi)$ . Then we have

$$\int_{a_\tau}^{b_\tau} \phi_\tau \left( \frac{\partial \phi_\tau}{\partial \psi} \right) F_4(\phi_\tau, \frac{\partial \phi_\tau}{\partial \psi}) d\psi = 0.$$

We prove this lemma in Appendix B.

By using this lemma, we can overcome the third difficulty. Indeed, if the  $C_{x_0}^1$  norm of  $\eta$  has a variation of size  $O(\varepsilon^2)$ , because of Lemma 3.5, its influence on 0th mode is only as big as it were  $O(\varepsilon^3)$ . We use  $\xi$  to cancel  $\Pi_0(R(\Upsilon, X_0, \eta, \Upsilon))$ . Obviously,  $\xi$  should have similar regularity as  $\Pi_0(R(\Upsilon, X_0, \eta, \Upsilon))$ . That is the reason we use the  $C_{x_0}^1$  norm of  $\xi$ . Note that the function  $\phi_\tau(\frac{\partial \phi_\tau}{\partial \psi}) F_4(\phi_\tau, \frac{\partial \phi_\tau}{\partial \psi})$  is in no sense an ‘‘odd’’ function. So it is non trivial that such a cancellation result holds.

### 3.4.2 The main theorem of 0th mode

Now we state the main theorem of 0th mode.

**Theorem 3.6.** *For fixed  $\tau_0 \in (0, \frac{1}{4})$ ,  $C_1, C_2 > 0$ , we can choose  $C_3, C_4 > 0$  and  $\delta_0 > 0$ , such that when  $\varepsilon \leq \delta_0$ , for every  $\|\xi\|_{C_{x_0}^1} \leq C_1 \varepsilon^2$ ,  $\|\mu\|_{C_\varepsilon^\alpha} \leq C_2$ , we can find unique*

$$|\omega_{\xi, \mu}| \leq C_3, |\phi(0)_{\xi, \mu} - \frac{1 - \sqrt{1 - 4\tau_0}}{2}| \leq C_4 \varepsilon^2$$

and  $\phi_{\xi, \mu, \omega_{\xi, \mu}, \phi(0)_{\xi, \mu}}(\psi)$  which solves (20) with  $\omega = \omega_{\xi, \mu}$ ,  $\phi(0) = \phi(0)_{\xi, \mu}$ .  
For  $\phi_{\xi, \mu, \omega_{\xi, \mu}, \phi(0)_{\xi, \mu}}(\psi)$  we have a  $C^1$  map

$$\tilde{\Phi} = \tilde{\Phi}_{\xi, \mu, \tau_0} : \Gamma \rightarrow \Gamma$$

such that

$$\begin{aligned} & |\phi_{\xi, \mu, \omega_{\xi, \mu}, \phi(0)_{\xi, \mu}}(\psi) - \phi_{\tau_0}(\tilde{\Phi}(\psi))| + \left| \frac{\partial \phi_{\xi, \mu, \omega_{\xi, \mu}, \phi(0)_{\xi, \mu}}(\psi)}{\partial \psi} - \frac{\partial \phi_{\tau_0}}{\partial \tilde{\Phi}(\psi)}(\tilde{\Phi}(\psi)) \right| \\ & \leq C(\tau_0, C_1, C_2) \varepsilon^2 \end{aligned}$$

and

$$\begin{cases} |\tilde{\Phi}(\psi) - \psi| & \leq C(\tau_0, C_1, C_2) \varepsilon, \\ |\tilde{\Phi}'(\psi) - 1| & \leq C(\tau_0, C_1, C_2) \varepsilon^2. \end{cases}$$

Moreover suppose  $\psi_i$  is the  $i$ th local minimum point of  $\phi_{\xi, \mu, \omega_{\xi, \mu}, \phi(0)_{\xi, \mu}}$ , then  $\tilde{\Phi}(\psi_i)$  is the  $i$ th local minimum point of  $\phi_{\tau_0}$ . In particular  $\tilde{\Phi}(0) = 0$ ,  $\tilde{\Phi}(L_\Gamma) = L_\Gamma$ .

Moreover for some  $C = C(\tau_0, C_1, C_2)$

$$\begin{cases} |\omega_{\xi_2, \mu_2} - \omega_{\xi_1, \mu_1}| & \leq C(\|\mu_2 - \mu_1\|_{C_\varepsilon^\alpha} + \frac{1}{\varepsilon} \|\xi_2 - \xi_1\|_{C_{x_0}^1}) \\ |\phi(0)_{\xi_2, \mu_2} - \phi(0)_{\xi_1, \mu_1}| & \leq C(\varepsilon^2 \|\mu_2 - \mu_1\|_{C_\varepsilon^\alpha} + \varepsilon \|\xi_2 - \xi_1\|_{C_{x_0}^1}) \end{cases}$$

$$\begin{aligned} & \|\phi_{\xi_2, \mu_2, \omega_{\xi_2, \mu_2}, \tau(0)_{\xi_2, \mu_2}}(\psi) - \phi_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi)\|_{C_\varepsilon^{2, \alpha}} \\ & \leq C(\varepsilon \|\mu_2 - \mu_1\|_{C_\varepsilon^\alpha} + \|\xi_2 - \xi_1\|_{C_{x_0}^1}). \end{aligned} \quad (23)$$

The proof of Theorem 3.6 is the most technical part of this paper. We prove it in six steps. In Step 1, we prove the local existence of the nonlinear ODE mentioned in (20); In Step 2, we prove the existence of the solution on the whole interval  $\psi \in [0, \frac{L_\Gamma}{\varepsilon}]$  without being concerned with boundary conditions (22). In Step 3, we do some basic estimates for the solutions. In Step 4, we analyze the linearization of the non linear ODE, which will be used in the next two steps. In Step 5, we adjust  $\omega$  and  $\tau(0)$  to match the boundary data on both sides of  $\psi = 0$  (or  $\psi = \frac{L_\Gamma}{\varepsilon}$ ). In Step 6, we derive the estimates in the theorem.



**Step 1. Local existence and uniqueness of ODE (20)** Choose  $A_1, A_2, B_1, B_2, K_1, K_2$  that only depend on  $\tau_0$  such that

$$0 < A_1 < A_2 < \frac{1 - \sqrt{1 - 4\tau(0)}}{2} < \frac{1 + \sqrt{1 - 4\tau(0)}}{2} < B_2 < B_1 < 1,$$

and  $0 < K_2 < K_1$  to be specified later. Define  $C_{A_i, B_i, K_i}^1([0, T]) = \{\phi(\psi) \in C^1([0, T]) | A_i \leq \phi(\psi) \leq B_i, |\phi_\psi| \leq K_i\}, i = 1, 2$ . We have

**Lemma 3.7.** *If  $\phi(\psi) \in C_{A_2, B_2, K_2}^1([0, T])$  solves the ODE system for  $T \geq 0$ . Then for some  $\delta > 0$  which only depends on  $\tau_0$ , this solution can be uniquely extended to  $\phi(\psi) \in C_{A_1, B_1, K_1}^1([0, T + \delta])$ .*

*Proof.* Denote  $\frac{\partial \phi}{\partial \psi}$  by  $\zeta$ . Then  $(\phi, \zeta)$  satisfies the following system

$$\begin{cases} \phi_\psi = \zeta, & \phi(0) = \frac{1 - \sqrt{1 - 4\tau(0)}}{2}, \\ \zeta_\psi = \phi^{-1}(1 + \zeta^2) - (2 + \rho)(1 + \zeta^2)^{\frac{3}{2}}, & \zeta(0) = 0, \end{cases}$$

The right hand side is uniformly bounded in  $(\phi, \zeta, \psi)$  and is a Lipschitz function with respect to  $(\phi, \zeta)$  in the domain

$$\begin{cases} |\zeta| < K_1 \\ A_1 < \phi < B_1 \\ 0 < \psi < \frac{L_\Gamma}{\varepsilon}. \end{cases}$$

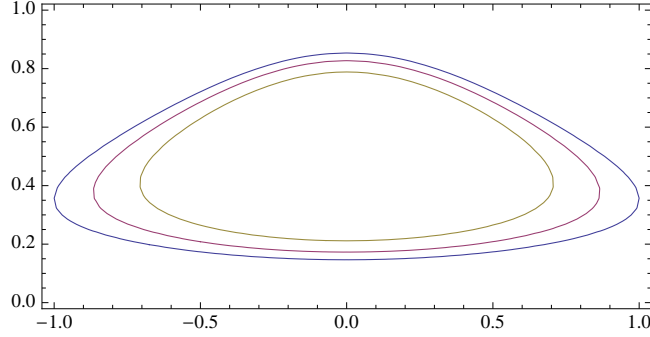
The  $C^0$  norm and Lipschitz constant of the right hand side only depend on  $\tau_0$ . So from standard theory of ODE we get the conclusion.  $\square$

**Step 2. Existence on  $[0, \frac{L_\Gamma}{\varepsilon}]$**  To get the existence on  $[0, \frac{L_\Gamma}{\varepsilon}]$ , we need do apriori estimates for the solution. The key to do this is the first integral  $\tau$ . Recall

$$\tau(\phi, \phi_\psi) = -\phi^2 + \frac{\phi}{\sqrt{1 + \phi_\psi^2}}.$$

We look at the phase space of this dynamical system. In the graph below the vertical direction represents  $\phi$  and the horizontal direction represents  $\zeta = \phi_\psi$ . The intermediate circle in the graph below represents the orbit of standard Delaunay surface with  $\tau = \tau_0 \in (0, \frac{1}{4})$ . We choose  $\delta_1(\tau_0) > 0$  sufficiently small. The outside circle represents Delaunay surface with  $\tau = \tau_0 - \delta_1 > 0$  while the inside one represents  $\tau = \tau_0 + \delta_1 < \frac{1}{4}$ . We can choose  $K_1 > K_2$  and  $K_2$  bigger than the maximal value of  $\zeta$  of the outside circle. We denote the closure of the domain between the outside and inside circle by  $A(\tau_0, \delta_1)$ . And we denote  $C_{A(\tau_0, \delta_1)}^1([0, T_0])$

$$\{\phi(\psi) \in C^1([0, T_0]) | (\phi, \phi_\psi) \in A(\tau_0, \delta_1)\}.$$



We have,

**Lemma 3.8.** (Existence on  $[0, \frac{L_\Gamma}{\varepsilon}]$ ) For fixed  $\xi, \mu, \omega, \tau(0)$  which satisfy (21), there is  $\tilde{\delta} > 0$  such that when  $\varepsilon < \tilde{\delta}$ , there is a unique solution  $\phi_{\xi, \mu, \omega, \tau(0)}(\psi) \in C^1_{A(\tau_0, \delta_1)}([0, \frac{1}{\varepsilon} L_\Gamma])$  to ODE mentioned in (20) (but may not satisfy boundary conditions (22)).

*Proof.* First one can choose  $\tilde{\delta}$  small that  $|\tau(0) - \tau_0| \leq C(\tau_0)\tilde{\delta} < \delta_1$ . Suppose the lemma were false. From Step 1, we assume  $T < \frac{L_\Gamma}{\varepsilon}$  is the maximal value that the solution  $\phi(\psi)$  can be extended in  $C^1_{A(\tau_0, \delta_1)}([0, T])$ . So  $(\phi(T), \zeta(T)) \in \partial A(\tau_0, \delta_1)$  must hold, that is to say  $\tau(\phi(T), \zeta(T)) = \tau_0 \pm \delta_1$ .

From easy calculations,

$$\frac{d\tau}{d\psi} = \rho(\psi)\phi\phi_\psi,$$

where  $\|\rho(\psi)\|_{C^0} \leq C(\varepsilon^2 + \varepsilon\|\xi\|_{C^0} + \varepsilon^3\|\mu\|_{C^0} + \varepsilon^3|\omega|)$ . So

$$\begin{aligned} & |\tau(\phi(T), \zeta(T)) - \tau(\phi(0), 0)| \\ &= \left| \int_0^T \rho(\psi)\phi\phi_\psi d\psi \right| \\ &\leq CT(\varepsilon^2 + \varepsilon\|\xi\|_{C^0} + \varepsilon^3\|\mu\|_{C^0} + \varepsilon^3|\omega|) \\ &\leq C(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) \\ &\leq C(\varepsilon + (C_1 + C_2 + C_3)\varepsilon^2). \end{aligned} \tag{24}$$

So we have

$$\begin{aligned} & |\tau(\phi(T), \zeta(T)) - \tau_0| \\ &\leq |\tau(\phi(T), \zeta(T)) - \tau(\phi(0), 0)| + |\tau(\phi(0), 0) - \tau_0| \\ &\leq C(\varepsilon + (C_1 + C_2 + C_3)\varepsilon^2) + C(\tau_0)\varepsilon. \end{aligned} \tag{25}$$

So we can choose  $\tilde{\delta}$  small enough, such that, when  $\varepsilon \leq \tilde{\delta}$

$$|\tau(\phi(T), \zeta(T)) - \tau_0| \leq \frac{\delta_1}{2}$$

which makes  $\tau(\phi(T), \zeta(T)) = \tau_0 \pm \delta_1$  impossible. So when  $\varepsilon \leq \tilde{\delta}$ , for any prescription of  $\xi, \mu, \omega, \tau(0)$  which satisfies (21), there is exactly one solution  $\phi_{\xi, \mu, \omega, \tau(0)}(\psi)$  for  $\psi \in [0, \frac{L_T}{\varepsilon}]$ .  $\square$

**Step 3. Estimates of the solution  $\phi_{\xi, \mu, \omega, \tau(0)}(\psi)$**  For simplicity we denote  $(\phi_{\xi, \mu, \omega, \tau(0)}(\psi), \frac{\partial \phi_{\xi, \mu, \omega, \tau(0)}(\psi)}{\partial \psi})$  as  $(\phi(\psi), \zeta(\psi))$ . From (24) we know,  $\tau(\phi(\psi), \zeta(\psi))$  will keep in  $C(C_1, C_2, C_3, \tau_0)\varepsilon$  neighborhood of  $\tau(0) = \tau(\phi(0), 0)$ . Actually we can improve this estimate to  $C\varepsilon^2$  neighborhood. This result comes from a simple observation. Note that

$$\begin{aligned} & |\tau(\phi(T), \zeta(T)) - \tau(\phi(0), 0)| \\ &= \left| \int_0^T \rho(\psi) \phi \phi_\psi d\psi \right| \\ &= \left| \int_0^T \phi \phi_\psi (-\varepsilon^2 F_1(\phi, \phi_\psi) \star \Pi_0(R_1) + \varepsilon F_4(\phi, \phi_\psi) \xi \right. \\ &\quad \left. + \varepsilon^3 \mu + \varepsilon^3 \omega \phi^{-1} \phi_\psi) d\psi \right|. \end{aligned} \tag{26}$$

Note that  $\varepsilon \xi = O(\varepsilon^3)$ ,  $\varepsilon^3 \mu = O(\varepsilon^3)$ ,  $\varepsilon^3 \omega \phi_\psi = O(\varepsilon^3)$ . The integral of these three terms is  $O(\varepsilon^2)$  because  $0 \leq T \leq \frac{L_T}{\varepsilon}$ . At first glance the integral of  $\phi \frac{\partial \phi}{\partial \psi} (-\varepsilon^2) F_1(\phi, \phi_\psi) \star \Pi_0(R_1)$  is of order  $O(\varepsilon)$ . However there is a cancellation property. Note that  $F_1(\phi, \phi_\psi) \phi \frac{\partial \phi}{\partial \psi}$  nearly has 0 average in one period (up to an error of order  $O(\varepsilon)$ ). Also we know  $\nabla_\psi R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) = O(\varepsilon)$ . So in one period the integral is only of size  $\varepsilon^3$ . When  $T$  is as large as  $\frac{L_T}{\varepsilon}$ , the integral is of size  $\varepsilon^2$ .

To be precise, we have

**Lemma 3.9.** *There exists  $C = C(C_1, C_2, C_3, \tau_0)$  which doesn't depend on  $\varepsilon$  such that for  $\psi \in [0, \frac{L_T}{\varepsilon}]$ ,  $(\phi(\psi), \zeta(\psi)) \in \text{Dom}(\tau(0), C\varepsilon^2)$ .*

*Proof.* We know that for each  $\psi \in [0, \frac{L_T}{\varepsilon}]$ ,  $|\tau(\phi(\psi), \zeta(\psi)) - \tau(\phi(0), 0)| \leq C(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|)$  from (24). Suppose  $\phi_{\tau(0)}(\psi)$  defines the standard Delaunay surface with parameter  $\tau \equiv \tau(\phi(0), 0)$  and  $\zeta_{\tau(0)}(\psi) = \frac{d}{d\psi} \phi_{\tau(0)}(\psi)$ . We assume the arc length parameter of the curve  $(\phi_{\tau(0)}(\psi), \zeta_{\tau(0)}(\psi))$  in the phase space is  $s_0$  ( $s_0$  is a multi-valued function on the curve  $(\phi_{\tau(0)}(\psi), \zeta_{\tau(0)}(\psi))$ ). We can extend  $s_0$  to a neighborhood of  $(\phi_{\tau(0)}(\psi), \zeta_{\tau(0)}(\psi))$  such that  $\frac{\partial}{\partial \tau} \perp \frac{\partial}{\partial s_0}$  holds everywhere in this neighborhood. We know that in this neighborhood

$$d\phi^2 + d\zeta^2 = \langle \partial_{s_0}, \partial_{s_0} \rangle ds_0^2 + \langle \partial_\tau, \partial_\tau \rangle d\tau^2$$

where  $\langle, \rangle$  denotes the inner product of the metric  $d\phi^2 + d\zeta^2$ . We know that  $\langle \partial_{s_0}, \partial_{s_0} \rangle = 1$  on  $(\phi_{\tau(0)}(\psi), \zeta_{\tau(0)}(\psi))$ . Regard  $(s_0, \tau)$  as new local coordinates of  $\text{Dom}(\tau(\phi(0), 0), C\varepsilon)$ . We define a continuous map  $\Phi = \Phi_{\xi, \mu, \omega, \tau(0)} : [0, \frac{L_T}{\varepsilon}] \rightarrow \mathbb{R}$  such that

$$s_0(\phi(\psi), \zeta(\psi)) = s_0(\phi_{\tau(0)}(\Phi(\psi)), \zeta_{\tau(0)}(\Phi(\psi))). \tag{27}$$

*Remark.* From (27) we see, if  $\zeta(\psi) = 0$ , then  $\zeta_{\tau(0)}(\Phi(\psi)) = 0$ . If  $\psi$  is a local minimum of  $\phi$ ,  $\Phi(\psi)$  is a local minimum of  $\phi_{\tau(0)}$

So we have

$$\begin{cases} |\phi(\psi) - \phi_{\tau(0)}(\Phi(\psi))| & \leq C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|), \\ |\zeta(\psi) - \zeta_{\tau(0)}(\Phi(\psi))| & \leq C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|). \end{cases} \quad (28)$$

**Lemma 3.10.**

$$\begin{cases} \Phi(0) = 0, \\ |\Phi(\psi) - \psi| \leq \frac{C(\tau_0)}{\varepsilon}(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|), \\ |\Phi'(\psi) - 1| \leq C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|). \end{cases} \quad (29)$$

In particular,  $\Phi : [0, \frac{L_\Gamma}{\varepsilon}] \rightarrow \Phi([0, \frac{L_\Gamma}{\varepsilon}])$  is invertible.

We prove Lemma 3.10 in Appendix C.

Now we continue to prove Lemma 3.9. Let  $[\tilde{\psi}_i, \tilde{\psi}_{i+1}]$  be the  $i$ th period of  $\phi_{\tau(0)}(\psi)$ . We assume  $\psi_i = \Phi^{-1}(\tilde{\psi}_i)$ . We know  $\phi_\psi(\psi_i) = 0$ . From (24)(28), we have

$$\begin{aligned} & \left| \int_{\psi_i}^{\psi_{i+1}} \phi \phi_\psi F_1(\phi, \phi_\psi) \star \Pi_0(R_1) d\psi \right| \\ & \leq \left| \int_{\tilde{\psi}_i}^{\tilde{\psi}_{i+1}} \phi_{\tau(0)} \frac{\partial \phi_{\tau(0)}}{\partial \psi} F_1(\phi_{\tau(0)}, \frac{\partial \phi_{\tau(0)}}{\partial \psi})|_\varsigma \star \Pi_0(R_1)|_{\Phi^{-1}(\varsigma)} d\varsigma \right| \\ & \quad + C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) \\ & = \left| \int_{\tilde{\psi}_i}^{\tilde{\psi}_{i+1}} \tilde{\chi}(\varsigma) \star \frac{\partial}{\partial \varsigma} \Pi_0(R_1)|_{\Phi^{-1}(\varsigma)} d\varsigma \right| \\ & \quad + C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) \\ & = C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|), \end{aligned}$$

where  $\tilde{\chi}(\varsigma)$  is the primitive and  $|\tilde{\chi}(s)|_{C^0} \leq C(\tau_0)\varepsilon$ . Now by dividing  $[0, \frac{L_\Gamma}{\varepsilon}]$  into “periods” of  $\phi(\psi)$ , it is easy to prove that, for any  $\bar{\psi} \in [0, \frac{L_\Gamma}{\varepsilon}]$

$$\begin{aligned} & \left| \int_0^{\bar{\psi}} \phi \phi_\psi \varepsilon^2 F_1(\phi, \phi_\psi) \star \Pi_0(R_1) d\psi \right| \\ & \leq C(\tau_0)\varepsilon(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|). \end{aligned}$$

So we can get better estimate for  $\tau$ ,

$$|\tau(\phi(T), \zeta(T)) - \tau(\phi(0), 0)| \leq C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) \quad (30)$$

for  $T \in [0, \frac{1}{\varepsilon}L_\Gamma]$ .

□

Now we can get better estimates for  $\Phi$  and  $\phi(\psi)$ , i.e.

$$\begin{cases} |\phi(\psi) - \phi_{\tau(0)}(\Phi(\psi))| & \leq C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|), \\ |\zeta(\psi) - \zeta_{\tau(0)}(\Phi(\psi))| & \leq C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|). \end{cases} \quad (31)$$

and

$$\begin{cases} \Phi(0) = 0, \\ |\Phi(\psi) - \psi| \leq \frac{C(\tau_0)}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|), \\ |\Phi'(\psi) - 1| \leq C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|). \end{cases} \quad (32)$$

**Corollary 3.11.**

$$\begin{aligned} & \|\phi(\psi) - \phi_{\tau(0)}(\psi)\|_{C_\varepsilon^{2,\alpha}} \\ & \leq C(\tau_0)\frac{1}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C_{x_0}^1} + \varepsilon^2\|\mu\|_{C_\varepsilon^\alpha} + \varepsilon^2|\omega|) \\ & \leq C(\tau_0)(1 + C_1 + C_2 + C_3)\varepsilon. \end{aligned}$$

*Proof.* From (31), (32) and  $\zeta_{\tau(0)}$ ,  $\frac{\partial^2 \phi_{\tau(0)}}{\partial \psi^2}$  are uniformly bounded by  $C(\tau_0)$ , we have

$$\begin{aligned} |\phi(\psi) - \phi_{\tau(0)}(\psi)| & \leq |\phi(\psi) - \phi_{\tau(0)}(\Phi(\psi))| + |\phi_{\tau(0)}(\Phi(\psi)) - \phi_{\tau(0)}(\psi)| \\ & \leq C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) \\ & \quad + \sup |\zeta_{\tau(0)}| \cdot |\Phi(\psi) - \psi| \\ & \leq \frac{C(\tau_0)}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) \end{aligned}$$

and in the same way

$$|\zeta(\psi) - \zeta_{\tau(0)}(\psi)| \leq \frac{C(\tau_0)}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|).$$

$|\phi^{-1} - \phi_{\tau(0)}^{-1}|$  also has similar estimates as  $\phi$  and  $\phi_{\tau(0)}$  has positive lower bounds. Finally from the ODE satisfied by  $\phi$  and  $\phi_{\tau(0)}$  and the two estimates above we have

$$\left\| \frac{\partial^2 \phi}{\partial \psi^2}(\psi) - \frac{\partial^2 \phi_{\tau(0)}}{\partial \psi^2}(\psi) \right\|_{C_\varepsilon^\alpha} \leq \frac{C(\tau_0)}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C_{x_0}^1} + \varepsilon^2\|\mu\|_{C_\varepsilon^\alpha} + \varepsilon^2|\omega|).$$

So the corollary is proved.  $\square$

**Corollary 3.12.** Suppose  $F(\theta_1, \theta_2)$  is a function with the following property. For any  $\tilde{C}_1(\tau_0), \tilde{C}_2(\tau_0)$ , there exists  $\tilde{C}_3(\tilde{C}_1, \tilde{C}_2)$  such that

$$\sup_{|\theta_1| \leq \tilde{C}_1, |\theta_2| \leq \tilde{C}_2} |F| + |\partial_{\theta_1} F| + |\partial_{\theta_2} F| \leq \tilde{C}_3.$$

Then for any

$$\|R(\psi)\|_{C_{x_0}^1(\Gamma)} \leq C$$

we have

$$\begin{aligned} & \left| \int_0^\psi F(\phi, \phi_\psi) R(\psi) d\psi - \int_0^{\Phi(\psi)} F(\phi_{\tau(0)}(\varsigma), \frac{\partial \phi_{\tau(0)}}{\partial \varsigma}) R(\varsigma) d\varsigma \right| \\ & \leq \frac{C(\tau_0)}{\varepsilon} (\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|) \|R\|_{C_{x_0}^1}. \end{aligned}$$

In particular, when  $R(\psi) \equiv 1$ ,

$$\begin{aligned} & \left| \int_0^\psi F(\phi, \phi_\psi) d\psi - \int_0^{\Phi(\psi)} F(\phi_{\tau(0)}(\varsigma), \frac{\partial \phi_{\tau(0)}}{\partial \varsigma}) d\varsigma \right| \\ & \leq \frac{C(\tau_0)}{\varepsilon} (\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|). \end{aligned}$$

*Proof.*

$$\begin{aligned} & \left| \int_0^\psi F(\phi, \phi_\psi) R(\psi) d\psi - \int_0^{\Phi(\psi)} F(\phi_{\tau(0)}(\varsigma), \frac{\partial \phi_{\tau(0)}}{\partial \varsigma}) R(\varsigma) d\varsigma \right| \\ & \leq \int_0^{\Phi(\psi)} \left| F(\phi, \phi_\psi)(\Phi^{-1}(\varsigma))(\Phi^{-1})' R(\Phi^{-1}(\varsigma)) - F(\phi_{\tau(0)}(\varsigma), \frac{\partial \phi_{\tau(0)}}{\partial \varsigma}) R(\varsigma) \right| d\varsigma \\ & \leq \frac{C(\tau_0)}{\varepsilon} (|\phi(\Phi^{-1}(\varsigma)) - \phi_{\tau(0)}(\varsigma)| |R| + |\phi_\psi(\Phi^{-1}(\varsigma)) - \frac{\partial \phi_{\tau(0)}}{\partial \varsigma}(\varsigma)| |R| \\ & \quad + |(\Phi^{-1})' - 1| |R| + |\partial_\psi R| |\Phi^{-1}(\varsigma) - \varsigma|) \\ & \leq \frac{C(\tau_0)}{\varepsilon} (\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|) \|R\|_{C_{x_0}^1}. \end{aligned}$$

□

**Step 4. The linearizations at  $\phi_{\xi, \mu, \omega, \tau(0)}(\psi)$**  In this step we analyze the linearizations of

$$\begin{cases} \frac{\partial^2 \phi}{\partial \psi^2} - \phi^{-1} (1 + (\frac{\partial \phi}{\partial \psi})^2) + (2 + \rho) (1 + (\frac{\partial \phi}{\partial \psi})^2)^{\frac{3}{2}} = 0 \\ \phi(0) = \frac{1 - \sqrt{1 - 4\tau(0)}}{2} \\ \phi'(0) = 0 \end{cases}$$

where

$$\begin{aligned} \rho = & -\varepsilon^2 F_1(\phi, \phi_\psi) \star \Pi_0(R_1) + \varepsilon F_4(\phi, \phi_\psi) \xi(x_0) \\ & + \varepsilon^3 \mu(\psi) + \varepsilon^3 \omega \phi^{-1} \phi_\psi \end{aligned}$$

where

$$\|\xi\|_{C_{x_0}^1} \leq C_1 \varepsilon^2, \|\mu\|_{C_\varepsilon^\alpha} \leq C_2, |\omega| \leq C_3, |\phi(0) - \frac{1 - \sqrt{1 - 4\tau_0}}{2}| \leq C(\tau_0) \varepsilon.$$

In Step 2, we have got a solution  $\phi(\psi) = \phi_{\xi, \mu, \omega, \tau(0)}(\psi)$ ,  $\psi \in [0, \frac{L_\Gamma}{\varepsilon}]$ . We can linearize the equation in four methods, i.e. to perturb  $\xi, \mu, \omega$  and  $\phi(0)$  (or  $\tau(0)$ ). For fixed  $\xi, \mu, \omega$  we suppose  $\phi_t$  is a class of solutions to

$$\frac{\partial^2 \phi}{\partial \psi^2} - \phi^{-1}(1 + (\frac{\partial \phi}{\partial \psi})^2) + (2 + \rho)(1 + (\frac{\partial \phi}{\partial \psi})^2)^{\frac{3}{2}} = 0,$$

(with different initial conditions) and  $\frac{d}{dt}\phi_t|_{t=0} = \beta(\psi)$ . Then we can calculate

$$\begin{aligned} & \mathcal{L}_{\xi, \mu, \omega, \tau(0)}\beta(\psi) \\ &= \frac{\partial^2 \beta}{\partial \psi^2} + (6(1 + \phi_\psi^2)^{\frac{1}{2}}\phi_\psi - 2\phi^{-1}\phi_\psi + \bar{F}_1)\frac{\partial \beta}{\partial \psi} + (\phi^{-2}(1 + \phi_\psi^2) + \bar{F}_2)\beta \\ &= 0 \end{aligned} \tag{33}$$

where  $\|\bar{F}_1\|_{C_\varepsilon^\alpha} + \|\bar{F}_2\|_{C_\varepsilon^\alpha} \leq \varepsilon^2(C + C_1 + C_2 + C_3)$ . First we analyze the fundamental solution to the linearized equation (33). Suppose  $\beta_1(\psi), \beta_2(\psi)$  satisfy

$$\mathcal{L}_{\xi, \mu, \omega, \tau(0)}\beta_i(\psi) = 0, \psi \in [0, \frac{L_\Gamma}{\varepsilon}]$$

and

$$\begin{pmatrix} \beta_1(0) & \beta_2(0) \\ \frac{\partial \beta_1}{\partial \psi}(0) & \frac{\partial \beta_2}{\partial \psi}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let  $R(\psi) = \begin{vmatrix} \beta_1(\psi) & \beta_2(\psi) \\ \beta_1'(\psi) & \beta_2'(\psi) \end{vmatrix}$ . Easy calculation implies

$$\frac{d}{d\psi} \log R(\psi) = -(6(1 + \phi_\psi^2)^{\frac{1}{2}}\phi_\psi - 2\phi^{-1}\phi_\psi + \bar{F}_1).$$

From Corollary 3.12, and the fact that  $6(1 + (\frac{\partial \phi_{\tau(0)}}{\partial \varsigma})^2)^{\frac{1}{2}}\frac{\partial \phi_{\tau(0)}}{\partial \varsigma} - 2\phi_{\tau(0)}^{-1}\frac{\partial \phi_{\tau(0)}}{\partial \varsigma}$  has 0 average in one period, we deduce that there exist  $C = C(\tau_0, C_1, C_2, C_3) > 0$  such that

$$e^{-C} \leq R(\psi) \leq e^C. \tag{34}$$

We define  $\mathcal{L}_{\tau(0)}$  as

$$\begin{aligned} & \mathcal{L}_{\tau(0)}\beta(\varsigma) \\ &= \frac{\partial^2 \beta}{\partial \varsigma^2} + (6(1 + (\frac{\partial \phi_{\tau(0)}}{\partial \varsigma})^2)^{\frac{1}{2}}\frac{\partial \phi_{\tau(0)}}{\partial \varsigma} - 2\phi_{\tau(0)}^{-1}\frac{\partial \phi_{\tau(0)}}{\partial \varsigma})\frac{\partial \beta}{\partial \varsigma} \\ & \quad + \phi_{\tau(0)}^{-2}(1 + (\frac{\partial \phi_{\tau(0)}}{\partial \varsigma})^2)\beta. \end{aligned}$$

By using Lemma 3.10 and Corollary 3.11 we can compare  $\mathcal{L}_{\xi, \mu, \omega, \tau(0)}$  with  $\mathcal{L}_{\tau(0)}$ .

We denote  $\psi_i, i = 0, 1, 2, \dots$  as the  $i$ th local minimum of  $\phi$  in  $[0, \frac{L_\Gamma}{\varepsilon}]$ . So  $\psi_0 = 0$  and  $[\psi_{i-1}, \psi_i]$  resembles the  $i$ th “period” of  $\phi(\psi)$ .  $\tilde{\psi}_i = \Phi(\psi_i)$  is the  $i$ th

local minimum of  $\phi_{\tau(0)}$ . When  $\lambda \in [0, 1]$ , let  $\psi = (1 - \lambda)\psi_{i-1} + \lambda\psi_i$ . From (31),(32) we have

$$\begin{aligned} & |\phi((1 - \lambda)\psi_{i-1} + \lambda\psi_i) - \phi_{\tau(0)}((1 - \lambda)\tilde{\psi}_{i-1} + \lambda\tilde{\psi}_i)| \\ & \leq |\phi(\psi) - \phi_{\tau(0)}(\Phi(\psi))| + |\phi_{\tau(0)}(\Phi(\psi)) - \phi_{\tau(0)}((1 - \lambda)\tilde{\psi}_{i-1} + \lambda\tilde{\psi}_i)| \\ & \leq C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|), \\ & |\zeta((1 - \lambda)\psi_{i-1} + \lambda\psi_i) - \zeta_{\tau(0)}((1 - \lambda)\tilde{\psi}_{i-1} + \lambda\tilde{\psi}_i)| \\ & \leq C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|). \end{aligned}$$

Suppose  $\beta_{1,i-1}, \beta_{2,i-1}$  solves

$$\mathcal{L}_{\xi,\mu,\omega,\tau(0)}\beta_{j,i-1}(\psi) = 0, \psi \in [\psi_{i-1}, \psi_i], j = 1, 2,$$

with

$$\begin{pmatrix} \beta_{1,i-1}(\psi_{i-1}) & \beta_{2,i-1}(\psi_{i-1}) \\ \beta'_{1,i-1}(\psi_{i-1}) & \beta'_{2,i-1}(\psi_{i-1}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

From the analysis on the Jacobi fields in [8], the two fundamental solutions of  $\mathcal{L}_{\tau(0)}W = 0$  can be expressed as

$$\begin{aligned} W_1(\varsigma) &= h_1(\tau(0))\varsigma \frac{\partial \phi_{\tau(0)}}{\partial \varsigma} + v_{\tau(0)}(\varsigma), \\ W_2(\varsigma) &= h_2(\tau(0)) \frac{\partial \phi_{\tau(0)}}{\partial \varsigma} \end{aligned}$$

where  $v_{\tau(0)}(\varsigma)$  is a periodic function whose period is the same as  $\phi_{\tau(0)}(\psi)$  and  $v_{\tau(0)}(0) = 1, v'_{\tau(0)}(0) = 0$ .  $h_i(\tau(0)), i = 1, 2$  are normalized constants and  $h_2(\tau(0)) \frac{\partial^2 \phi_{\tau(0)}}{\partial \varsigma^2}(0) = 1$ . We see

$$\begin{pmatrix} W_1(0) & W_2(0) \\ \frac{\partial W_1}{\partial \psi_0}(0) & \frac{\partial W_2}{\partial \psi_0}(0) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{pmatrix} W_1(\tilde{\psi}_1) & W_2(\tilde{\psi}_1) \\ \frac{\partial W_1}{\partial \varsigma}(\tilde{\psi}_1) & \frac{\partial W_2}{\partial \varsigma}(\tilde{\psi}_1) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \kappa & 1 \end{pmatrix}$$

where  $\kappa(\tau(0)) \neq 0$  is a constant which only depends on  $\tau(0)$ .

Suppose  $W_{1,i-1}, W_{2,i-1}$  solves

$$\mathcal{L}_{\tau(0)}W_{j,i-1} = 0, \psi \in [\tilde{\psi}_{i-1}, \tilde{\psi}_i], j = 1, 2,$$

with

$$\begin{pmatrix} W_{1,i-1}(\tilde{\psi}_{i-1}) & W_{2,i-1}(\tilde{\psi}_{i-1}) \\ W'_{1,i-1}(\tilde{\psi}_{i-1}) & W'_{2,i-1}(\tilde{\psi}_{i-1}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is obvious that  $\tilde{\psi}_i = i\tilde{\psi}_1$  and  $W_{j,i}(\psi) = W_j(\psi - \tilde{\psi}_i)$ .



Comparing the coefficients of  $\mathcal{L}_{\xi, \mu, \omega, \tau(0)}$  with that of  $\mathcal{L}_{\tau(0)}$  we have

$$\begin{cases} |\beta_{j,i-1}((1-\lambda)\psi_{i-1} + \lambda\psi_i) - W_j(\lambda(\tilde{\psi}_i - \tilde{\psi}_{i-1}))| & \leq C(\tau_0)(1 + C_1 + C_2 + C_3)\varepsilon^2, \\ |\beta'_{j,i-1}((1-\lambda)\psi_{i-1} + \lambda\psi_i) - W'_j(\lambda(\tilde{\psi}_i - \tilde{\psi}_{i-1}))| & \leq C(\tau_0)(1 + C_1 + C_2 + C_3)\varepsilon^2, \end{cases} \quad (35)$$

which implies the following lemma,

**Lemma 3.13.**

$$\begin{aligned} & \|\beta_{1,i-1}(\psi) - (h_1(\tau(0))(\psi - \psi_i) \frac{\partial \phi}{\partial \psi} + v_1(\psi))\|_{C_\varepsilon^1([\psi_{i-1}, \psi_i])} \\ & + \|\beta_{2,i-1}(\psi) - h_2(\tau(0)) \frac{\partial \phi}{\partial \psi}\|_{C_\varepsilon^1([\psi_{i-1}, \psi_i])} \\ & \leq C(\tau_0)(1 + C_1 + C_2 + C_3)\varepsilon^2, \end{aligned}$$

where  $v_1((1-\lambda)\psi_{i-1} + \lambda\psi_i) = v_{\tau(0)}(\lambda(\tilde{\psi}_i - \tilde{\psi}_{i-1}))$ .

So from Lemma 3.13 we have

$$\begin{pmatrix} \beta_{1,i-1}(\psi_i) & \beta_{2,i-1}(\psi_i) \\ \frac{\partial \beta_{1,i-1}}{\partial \psi}(\psi_i) & \frac{\partial \beta_{2,i-1}}{\partial \psi}(\psi_i) \end{pmatrix} = \begin{pmatrix} 1 + e_{11}^i & e_{12}^i \\ \kappa + e_{21}^i & 1 + e_{22}^i \end{pmatrix}, \quad (36)$$

where  $|e_{jk}^i| \leq C(\tau_0)(1 + C_1 + C_2 + C_3)\varepsilon^2$ .

From the theory of linear ODE, we know

$$\begin{aligned} \begin{pmatrix} \beta_1(\psi_i) & \beta_2(\psi_i) \\ \frac{\partial \beta_1}{\partial \psi}(\psi_i) & \frac{\partial \beta_2}{\partial \psi}(\psi_i) \end{pmatrix} &= \begin{pmatrix} 1 + e_{11}^i & e_{12}^i \\ \kappa + e_{21}^i & 1 + e_{22}^i \end{pmatrix} \cdots \begin{pmatrix} 1 + e_{11}^1 & e_{12}^1 \\ \kappa + e_{21}^1 & 1 + e_{22}^1 \end{pmatrix} \\ &= \begin{pmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{pmatrix}. \end{aligned}$$

By analysis of the matrix, we can prove

**Lemma 3.14.**

$$\begin{aligned} & \exp(-C(\tau_0)(1 + C_1 + C_2 + C_3)) \\ & \leq A_{11}^i + A_{22}^i + \varepsilon A_{21}^i \\ & \leq \exp C(\tau_0)(1 + C_1 + C_2 + C_3), \\ & |A_{12}^i| \\ & \leq C(\tau_0, C_1, C_2, C_3)\varepsilon \exp C(\tau_0)(1 + C_1 + C_2 + C_3). \end{aligned}$$

We will prove this result in Appendix D. So from Lemma 3.13 we have

**Lemma 3.15.** For  $\xi, \mu, \omega, \tau(0)$  which satisfy (21), there is  $\delta > 0$  such that when  $0 < \varepsilon < \delta$ , in the interval  $[\psi_i, \psi_{i+1}]$ , we have

$$\begin{aligned} & \|\beta_1(\psi) - (A_{11}^i[h_1(\tau(0))(\psi - \psi_i)\frac{\partial\phi}{\partial\psi} + v_i(\psi)] + A_{21}^i h_2(\tau(0))\frac{\partial\phi}{\partial\psi})\|_{C_\varepsilon^1} \\ & \leq C(\tau_0, C_1, C_2, C_3)\varepsilon, \\ & \|\beta_2(\psi) - (A_{12}^i[h_1(\tau(0))(\psi - \psi_i)\frac{\partial\phi}{\partial\psi} + v_i(\psi)] + A_{22}^i h_2(\tau(0))\frac{\partial\phi}{\partial\psi})\|_{C_\varepsilon^1} \\ & \leq C(\tau_0, C_1, C_2, C_3)\varepsilon^2. \end{aligned}$$

where  $v_i((1 - \lambda)\psi_i + \lambda\psi_{i+1}) = v_{\tau(0)}((1 - \lambda)\Phi(\psi_i) + \lambda\Phi(\psi_{i+1}))$  for  $\lambda \in [0, 1]$  and  $A_{kl}^i$  satisfy the inequalities in Lemma 3.14.

In particular,  $\beta_1$  has linear growth and  $\beta_2$  is bounded.

We denote

$$\begin{cases} \frac{d}{dt}\phi_{\xi, \mu+t\Delta\mu, \omega, \tau(0)}(\psi)|_{t=0} &= \beta_\mu(\psi), \\ \frac{d}{dt}\phi_{\xi+t\Delta\xi, \mu, \omega, \tau(0)}(\psi)|_{t=0} &= \beta_\xi(\psi), \\ \frac{d}{dt}\phi_{\xi, \mu, \omega+t, \tau(0)}(\psi)|_{t=0} &= \beta_\omega(\psi). \end{cases}$$

We have that the following estimates

**Lemma 3.16.**

$$\|\beta_\mu(\psi)\|_{C_\varepsilon^1} \leq C(\tau_0, C_1, C_2, C_3)\varepsilon\|\Delta\mu\|_{C^0}, \quad (37)$$

$$\|\beta_\xi(\psi)\|_{C_\varepsilon^1} \leq C(\tau_0, C_1, C_2, C_3)\|\Delta\xi\|_{C_{x_0}^1}, \quad (38)$$

$$\|\beta_\omega(\psi)\|_{C_\varepsilon^1} \leq C(\tau_0, C_1, C_2, C_3)\varepsilon. \quad (39)$$

Moreover if we consider

$$\frac{\partial(\Delta\tau, \zeta(\frac{L_\Gamma}{\varepsilon}))}{\partial(\omega, \phi(0))}$$

where  $\Delta\tau = \tau(\frac{L_\Gamma}{\varepsilon}) - \tau(0)$  and  $\zeta = \phi_\psi$ . We can get

$$\begin{cases} \frac{\partial\Delta\tau}{\partial\omega} & \geq C_5(\tau_0)\varepsilon^2 \\ \left|\frac{\partial\Delta\tau}{\partial\phi(0)}\right| & \leq K_1(\tau_0, C_1, C_2, C_3)\varepsilon \\ \left|\frac{\partial\zeta(\frac{L_\Gamma}{\varepsilon})}{\partial\omega}\right| & = |\beta'_\omega(\frac{L_\Gamma}{\varepsilon})| \leq K_2(\tau_0, C_1, C_2, C_3)\varepsilon \\ \frac{\partial\zeta(\frac{L_\Gamma}{\varepsilon})}{\partial\phi(0)} & = \beta'_1(\frac{L_\Gamma}{\varepsilon}) \geq \exp(-C(\tau_0)(1 + C_1 + C_2 + C_3))\frac{1}{\varepsilon}. \end{cases} \quad (40)$$

We will prove this lemma in Appendix E.

### Step 5. Match the boundary value

**Lemma 3.17.** For  $C_1, C_2 > 0$  we can choose  $C_3, C_4$  and  $\delta$  such that when  $0 < \varepsilon < \delta$ , if  $\|\xi\|_{C_{x_0}^1} \leq C_1\varepsilon^2$  and  $\|\mu\|_{C_\varepsilon^\alpha} \leq C_2$  there are unique  $\omega_{\xi, \mu}$  and  $\phi(0)_{\xi, \mu}$  (or  $\tau(0)_{\xi, \mu}$ ) such that

$$\begin{cases} \phi_{\xi, \mu, \omega_{\xi, \mu}, \phi(0)_{\xi, \mu}}(0) = \phi_{\xi, \mu, \omega_{\xi, \mu}, \phi(0)_{\xi, \mu}}(\frac{L_\Gamma}{\varepsilon}) = \phi(0)_{\xi, \mu}, \\ \phi'_{\xi, \mu, \omega_{\xi, \mu}, \phi(0)_{\xi, \mu}}(0) = \phi'_{\xi, \mu, \omega_{\xi, \mu}, \phi(0)_{\xi, \mu}}(\frac{L_\Gamma}{\varepsilon}) = 0 \end{cases}$$

and

$$\begin{aligned} |\omega_{\xi,\mu}| &\leq C_3, \\ |\phi(0)_{\xi,\mu} - \frac{1 - \sqrt{1 - 4\tau_0}}{2}| &\leq C_4 \varepsilon^2. \end{aligned}$$

*Proof.* We begin with  $\omega = 0, \tau(0) = \tau_0$ . From (30) and Corollary 3.11 we have for a particular  $\tilde{C} = \tilde{C}(\tau_0)$

$$\begin{aligned} |\Delta\tau| = |\tau(\frac{L_\Gamma}{\varepsilon}) - \tau(0)| &\leq \tilde{C}(\tau_0)\varepsilon^2(1 + C_1 + C_2) \\ |\zeta(\frac{L_\Gamma}{\varepsilon})| &\leq \tilde{C}(\tau_0)\varepsilon(1 + C_1 + C_2). \end{aligned}$$

We choose

$$C_3 = \frac{4\tilde{C}(\tau_0)(1 + C_1 + C_2)}{C_5(\tau_0)}$$

and

$$C_4 = 4\tilde{K}_2 C_5^{-1} \tilde{C}(\tau_0)(1 + C_1 + C_2) \exp(C(\tau_0)(1 + C_1 + C_2 + C_3))$$

where  $\tilde{K}_2 = \tilde{K}_2(\tau_0, C_1, C_2, C_3, C_5) = K_2(\tau_0, C_1, C_2, C_3) + C_5$ . We choose

$$0 < \varepsilon \leq C_6 = \min\left\{\frac{C(\tau_0)}{C_4}, \frac{C_5 \exp(-C(\tau_0)(1 + C_1 + C_2 + C_3))}{2|K_1 \tilde{K}_2| + 1}\right\}$$

and also  $\varepsilon$  is small enough so that we can go through all the analysis in the above modes and steps. Note that first  $C_4 \varepsilon^2 \leq C(\tau_0)\varepsilon$ , so one can work with a uniform constant  $C_5(\tau_0) > 0$  such that

$$\frac{\partial \Delta\tau}{\partial \omega} \geq C_5 \varepsilon^2.$$

First we prove the uniqueness. For  $\omega_1, \omega_2 \in [-C_3, C_3], \phi(0)_1, \phi(0)_2 \in [\frac{1 - \sqrt{1 - 4\tau_0}}{2} - C_4 \varepsilon^2, \frac{1 - \sqrt{1 - 4\tau_0}}{2} + C_4 \varepsilon^2]$ , if  $\Delta\tau$  and  $\zeta(\frac{L_\Gamma}{\varepsilon})$  takes the same value at  $(\omega_1, \phi(0)_1)$  and  $(\omega_2, \phi(0)_2)$  then there exists  $\omega_3, \omega_4$  which lie between  $\omega_1, \omega_2$  and  $\phi(0)_3, \phi(0)_4$  lie between  $\phi(0)_1, \phi(0)_2$  such that

$$\begin{bmatrix} \frac{\partial \Delta\tau}{\partial \omega}|_{(\omega_3, \phi(0)_1)} & \frac{\partial \Delta\tau}{\partial \phi(0)}|_{(\omega_2, \phi(0)_3)} \\ \frac{\partial \zeta(\frac{L_\Gamma}{\varepsilon})}{\partial \omega}|_{(\omega_4, \phi(0)_1)} & \frac{\partial \zeta(\frac{L_\Gamma}{\varepsilon})}{\partial \phi(0)}|_{(\omega_2, \phi(0)_4)} \end{bmatrix} \begin{bmatrix} \omega_1 - \omega_2 \\ \phi(0)_1 - \phi(0)_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

From estimate (40), the matrix  $\begin{bmatrix} \frac{\partial \Delta\tau}{\partial \omega}|_{(\omega_3, \phi(0)_1)} & \frac{\partial \Delta\tau}{\partial \phi(0)}|_{(\omega_2, \phi(0)_3)} \\ \frac{\partial \zeta(\frac{L_\Gamma}{\varepsilon})}{\partial \omega}|_{(\omega_4, \phi(0)_1)} & \frac{\partial \zeta(\frac{L_\Gamma}{\varepsilon})}{\partial \phi(0)}|_{(\omega_2, \phi(0)_4)} \end{bmatrix}$  is invertible as long as  $\varepsilon \leq C_6$ . So  $\omega_1 = \omega_2$  and  $\phi(0)_1 = \phi(0)_2$ . We have proved the uniqueness.

For the existence first we perturb  $\omega$  within  $\frac{\tilde{C}(1 + C_1 + C_2)}{C_5}$  such that  $\Delta\tau = 0$ . Then  $\zeta(\frac{L_\Gamma}{\varepsilon})$  will change no more than  $K_2 \frac{\tilde{C}(1 + C_1 + C_2)}{C_5} \varepsilon$ . Then we perturb  $\phi(0)$  within

$$\tilde{K}_2 C_5^{-1} \tilde{C}(1 + C_1 + C_2) \exp(C(\tau_0)(1 + C_1 + C_2 + C_3)) \varepsilon^2$$

such that  $\zeta(\frac{L_\Gamma}{\varepsilon}) = 0$ . Then  $\Delta\tau$  will change no more than  $K_1\tilde{K}_2C_5^{-1}\tilde{C}(1+C_1+C_2)\exp(C(\tau_0)(1+C_1+C_2+C_3))\varepsilon^3$ . From

$$\varepsilon \leq \frac{C_5 \exp(-C(\tau_0)(1+C_1+C_2+C_3))}{2|K_1\tilde{K}_2|+1}$$

we know

$$\begin{aligned} & K_1\tilde{K}_2C_5^{-1}\tilde{C}(1+C_1+C_2)\exp(C(\tau_0)(1+C_1+C_2+C_3))\varepsilon^3 \\ & \leq \frac{1}{2}\tilde{C}(1+C_1+C_2)\varepsilon^2. \end{aligned}$$

Then by an iteration argument, we have, there exists  $|\omega_{\xi,\mu}| \leq C_3$  and  $|\phi(0)_{\xi,\mu} - \frac{1-\sqrt{1-4\tau_0}}{2}| \leq C_4\varepsilon^2$  such that  $\Delta\tau = 0$  and  $\zeta(\frac{L_\Gamma}{\varepsilon}) = 0$ . From  $|\phi_{\xi,\mu,\omega_{\xi,\mu},\tau(0)_{\xi,\mu}}(\frac{L_\Gamma}{\varepsilon}) - \phi_{\xi,\mu,\omega_{\xi,\mu},\tau(0)_{\xi,\mu}}(0)| \leq C\varepsilon$ , we know  $\phi_{\xi,\mu,\omega_{\xi,\mu},\tau(0)_{\xi,\mu}}(\frac{L_\Gamma}{\varepsilon}) = \phi_{\xi,\mu,\omega_{\xi,\mu},\tau(0)_{\xi,\mu}}(0)$ .  $\square$

From Corollary 3.11 and Lemma 3.17 we have

**Corollary 3.18.** *For given  $C_1, C_2$  and  $\tau_0$  there is  $C = C(\tau_0, C_1, C_2)$  such that*

$$\|\phi_{\xi,\mu}(\psi) - \phi_{\tau_0}(\psi)\|_{C_\varepsilon^{2,\alpha}} \leq C\varepsilon.$$

From (31) and Lemma 3.17 we have

**Corollary 3.19.** *For  $\phi_{\xi,\mu,\omega_{\xi,\mu},\phi(0)_{\xi,\mu}}(\psi)$  we have a  $C^1$  map*

$$\tilde{\Phi} = \tilde{\Phi}_{\xi,\mu,\tau_0} : \Gamma \rightarrow \Gamma$$

such that

$$\begin{aligned} & |\phi_{\xi,\mu,\omega_{\xi,\mu},\phi(0)_{\xi,\mu}}(\psi) - \phi_{\tau_0}(\tilde{\Phi}(\psi))| + |\zeta_{\xi,\mu,\omega_{\xi,\mu},\phi(0)_{\xi,\mu}}(\psi) - \zeta_{\tau_0}(\tilde{\Phi}(\psi))| \\ & \leq C(\tau_0, C_1, C_2)\varepsilon^2 \end{aligned}$$

and

$$\begin{cases} |\tilde{\Phi}(\psi) - \psi| & \leq C(\tau_0, C_1, C_2)\varepsilon, \\ |\tilde{\Phi}'(\psi) - 1| & \leq C(\tau_0, C_1, C_2)\varepsilon^2. \end{cases}$$

Moreover suppose  $\psi_i$  is the  $i$ th local minimum point of  $\phi_{\xi,\mu,\omega_{\xi,\mu},\phi(0)_{\xi,\mu}}$ , then  $\tilde{\Phi}(\psi_i)$  is the  $i$ th local minimum point of  $\phi_{\tau_0}$ . In particular  $\tilde{\Phi}(0) = 0, \tilde{\Phi}(L_\Gamma) = L_\Gamma$ .

**Step 6. Main estimates of 0th mode** For

$$\|\xi_1\|_{C_{x_0}^1}, \|\xi_2\|_{C_{x_0}^1} \leq C_1\varepsilon^2, \|\mu_1\|_{C_\varepsilon^\alpha}, \|\mu_2\|_{C_\varepsilon^\alpha} \leq C_2,$$

we can get unique  $\phi_{\xi_1,\mu_1,\omega_{\xi_1,\mu_1},\tau(0)_{\xi_1,\mu_1}}(\psi)$  and  $\phi_{\xi_2,\mu_2,\omega_{\xi_2,\mu_2},\tau(0)_{\xi_2,\mu_2}}(\psi)$  satisfying

$$|\omega_{\xi_1,\mu_1}|, |\omega_{\xi_2,\mu_2}| \leq C_3$$

and

$$|\phi(0)_{\xi_1, \mu_1} - \frac{1 - \sqrt{1 - 4\tau_0}}{2}|, |\phi(0)_{\xi_2, \mu_2} - \frac{1 - \sqrt{1 - 4\tau_0}}{2}| \leq C_4 \varepsilon^2$$

such that for  $i = 1, 2$ ,

$$\begin{cases} \phi_{\xi_i, \mu_i, \omega_{\xi_i, \mu_i}, \tau(0)_{\xi_i, \mu_i}}(0) &= \phi_{\xi_i, \mu_i, \omega_{\xi_i, \mu_i}, \tau(0)_{\xi_i, \mu_i}}(\frac{L_\Gamma}{\varepsilon}) = \phi(0)_{\xi_i, \mu_i}, \\ \zeta_{\xi_i, \mu_i, \omega_{\xi_i, \mu_i}, \tau(0)_{\xi_i, \mu_i}}(0) &= \zeta_{\xi_i, \mu_i, \omega_{\xi_i, \mu_i}, \tau(0)_{\xi_i, \mu_i}}(\frac{L_\Gamma}{\varepsilon}) = 0. \end{cases}$$

First

$$\begin{aligned} & \|\phi_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi) - \phi_{\xi_2, \mu_2, \omega_{\xi_2, \mu_2}, \tau(0)_{\xi_2, \mu_2}}(\psi)\|_{C_\varepsilon^1} \\ & \leq \|\phi_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi) - \phi_{\xi_2, \mu_2, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi)\|_{C_\varepsilon^1} \\ & \quad + \|\phi_{\xi_2, \mu_2, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi) - \phi_{\xi_2, \mu_2, \omega_{\xi_2, \mu_2}, \tau(0)_{\xi_2, \mu_2}}(\psi)\|_{C_\varepsilon^1}. \end{aligned}$$

Note that from (37) and (38) we have

$$\begin{aligned} & \|\phi_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi) - \phi_{\xi_2, \mu_2, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi)\|_{C_\varepsilon^1} \\ & \leq \int_0^1 \left\| \frac{d}{dt} \phi_{t\xi_1 + (1-t)\xi_2, t\mu_1 + (1-t)\mu_2, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi) \right\|_{C_\varepsilon^1} dt \\ & \quad + \int_0^1 \left\| \frac{d}{dt} \phi_{\xi_2, t\mu_1 + (1-t)\mu_2, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi) \right\|_{C_\varepsilon^1} dt \\ & \leq C\varepsilon \|\mu_2 - \mu_1\|_{C_{x_0}^\alpha} + C\|\xi_2 - \xi_1\|_{C_{x_0}^1}. \end{aligned}$$

By comparing  $\phi_{\xi_2, \mu_2, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi)$  with  $\phi_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi)$ , we have for  $\phi_{\xi_2, \mu_2, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi)$

$$\begin{aligned} |\Delta\tau| & \leq C(\tau_0, C_1, C_2, C_3)(\varepsilon^2 \|\mu_2 - \mu_1\|_{C_\varepsilon^\alpha} + \varepsilon \|\xi_2 - \xi_1\|_{C_{x_0}^1}), \\ |\zeta(\frac{L_\Gamma}{\varepsilon})| & \leq C(\tau_0, C_1, C_2, C_3)(\varepsilon \|\mu_2 - \mu_1\|_{C_\varepsilon^\alpha} + \|\xi_2 - \xi_1\|_{C_{x_0}^1}). \end{aligned}$$

If we denote

$$\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} = \frac{\partial(\Delta\tau, \zeta(\frac{L_\Gamma}{\varepsilon}))}{\partial(\omega, \phi(0))}^{-1},$$

we can get

$$\begin{aligned} |B_{11}| & \leq 2C_5^{-1} \varepsilon^{-2} \\ |B_{22}| & \leq 2 \exp(C(\tau_0)(1 + C_1 + C_2 + C_3)) \varepsilon \\ |B_{12}| & \leq 2K_1 C_5^{-1} \exp(C(\tau_0)(1 + C_1 + C_2 + C_3)) \\ |B_{21}| & \leq 2K_2 C_5^{-1} \exp(C(\tau_0)(1 + C_1 + C_2 + C_3)). \end{aligned}$$

So we know for some  $C = C(\tau_0, C_1, C_2, C_3)$

$$\begin{cases} |\omega_{\xi_2, \mu_2} - \omega_{\xi_1, \mu_1}| & \leq \frac{C}{\varepsilon^2} (\varepsilon^2 \|\mu_2 - \mu_1\|_{C_\varepsilon^\alpha} + \varepsilon \|\xi_2 - \xi_1\|_{C_{x_0}^1}) \\ |\phi(0)_{\xi_2, \mu_2} - \phi(0)_{\xi_1, \mu_1}| & \leq C(\varepsilon^2 \|\mu_2 - \mu_1\|_{C_\varepsilon^\alpha} + \varepsilon \|\xi_2 - \xi_1\|_{C_{x_0}^1}) \end{cases}$$

So from (39) and Lemma 3.15 we have

$$\begin{aligned}
& |\phi_{\xi_2, \mu_2, \omega_{\xi_2, \mu_2}, \tau(0)_{\xi_2, \mu_2}}(\psi) - \phi_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi)| \\
& \leq \frac{C}{\varepsilon} (\varepsilon^2 \|\mu_2 - \mu_1\|_{C_\varepsilon^\alpha} + \varepsilon \|\xi_2 - \xi_1\|_{C_{x_0}^1}) \\
& |\zeta_{\xi_2, \mu_2, \omega_{\xi_2, \mu_2}, \tau(0)_{\xi_2, \mu_2}}(\psi) - \zeta_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi)| \\
& \leq \frac{C}{\varepsilon} (\varepsilon^2 \|\mu_2 - \mu_1\|_{C_\varepsilon^\alpha} + \varepsilon \|\xi_2 - \xi_1\|_{C_{x_0}^1})
\end{aligned}$$

And from the ODE satisfied by  $\phi$ , we can get high order estimates, i.e.

$$\begin{aligned}
& \|\phi_{\xi_2, \mu_2, \omega_{\xi_2, \mu_2}, \tau(0)_{\xi_2, \mu_2}}(\psi) - \phi_{\xi_1, \mu_1, \omega_{\xi_1, \mu_1}, \tau(0)_{\xi_1, \mu_1}}(\psi)\|_{C_\varepsilon^{2, \alpha}} \\
& \leq \frac{C}{\varepsilon} (\varepsilon^2 \|\mu_2 - \mu_1\|_{C_\varepsilon^\alpha} + \varepsilon \|\xi_2 - \xi_1\|_{C_{x_0}^1}).
\end{aligned}$$

At last note  $C_3 = C_3(\tau_0, C_1, C_2)$ . So  $C(\tau_0, C_1, C_2, C_3) = C(\tau_0, C_1, C_2)$ .

## 4 The existence of CMC surfaces

### 4.1 A fixed point argument

In 0th mode, if we prescribe  $\xi \in C_{x_0}^1$ ,  $\mu \in C_\varepsilon^\alpha$ ,  $\omega_{\xi, \mu}$  and  $\phi(0)_{\xi, \mu}$  in  $\rho$ , we can get  $\phi_{\xi, \mu, \omega_{\xi, \mu}, \phi(0)_{\xi, \mu}}$ , which we denote by  $\phi_{\xi, \mu}$  for short. We define parameter  $s$  using  $\dot{\psi} = \frac{d\psi}{ds} = \phi_{\xi, \mu}^2 + \tau_{\xi, \mu}$  where  $\tau_{\xi, \mu} = -\phi_{\xi, \mu}^2 + \frac{\phi_{\xi, \mu}}{\sqrt{1 + \zeta_{\xi, \mu}^2}}$ . From (20) we know

$$\dot{\phi}_{\xi, \mu}^2 + (\phi_{\xi, \mu}^2 + \tau_{\xi, \mu})^2 = \phi_{\xi, \mu}^2.$$

And further we have

$$\begin{aligned}
\ddot{\phi}_{\xi, \mu} &= \phi_{\xi, \mu} - (2 + \rho)\phi_{\xi, \mu}(\phi_{\xi, \mu}^2 + \tau_{\xi, \mu}), \\
\ddot{\psi}_{\xi, \mu} &= \phi_{\xi, \mu}\dot{\phi}_{\xi, \mu}(2 + \rho).
\end{aligned}$$

Follow the calculations of mean curvature in Appendix A and we can get  $H(\mathcal{D}_{\phi_{\xi, \mu}, p_0, \varepsilon}(w, \eta))$ . We substitute

$$\frac{\dot{\phi}_{\xi, \mu}\ddot{\psi}}{\phi_{\xi, \mu}} - \frac{\ddot{\phi}_{\xi, \mu}\dot{\psi}}{\phi_{\xi, \mu}} = \phi_{\xi, \mu}^2 - \tau_{\xi, \mu} + \phi_{\xi, \mu}^2\rho$$

for (50). Noticing every place where  $\ddot{\phi}, \ddot{\psi}$  appear, we can get

$$\begin{aligned}
& H(\mathcal{D}_{\phi_{\xi, \mu}, p_0, \varepsilon}(\tilde{w}, \eta)) \\
& = \frac{2}{\varepsilon} + \frac{\rho}{\varepsilon} + \frac{1}{\varepsilon}\tilde{\mathcal{L}}_{\xi, \mu}\tilde{w} + \langle \mathcal{J}_{\xi, \mu}\eta, \Upsilon \rangle \\
& \quad + \varepsilon(F_1(\phi_{\xi, \mu}, \zeta_{\xi, \mu}) \star R_1 + F_2(\phi_{\xi, \mu}, \zeta_{\xi, \mu}) \star R_2) + E_{\xi, \mu} \\
& \quad + F_3(\phi_{\xi, \mu}, \zeta_{\xi, \mu}) \star R_3(\eta) + T_{\xi, \mu}(\tilde{w}, \eta) \\
& \quad + \varepsilon^{-1}\rho L_{\xi, \mu}^3(\tilde{w}, \eta) + \varepsilon^{-1}\rho Q_{\xi, \mu}^3(\tilde{w}, \eta) + \varepsilon L_{\xi, \mu}^4(\rho),
\end{aligned} \tag{41}$$

where

$$\begin{aligned}
& \tilde{\mathcal{L}}_{\xi,\mu} \tilde{w} \\
&= -\frac{\dot{\psi}_{\xi,\mu}}{\phi_{\xi,\mu}^3} \left( \frac{\partial^2}{\partial s^2} + \frac{\partial^2}{\partial \theta^2} \right) \tilde{w} - (\phi_{\xi,\mu}^2 - \tau_{\xi,\mu}) \frac{\dot{\phi}_{\xi,\mu}}{\phi_{\xi,\mu}^4} \frac{\partial w}{\partial s} - \frac{\dot{\psi}_{\xi,\mu}}{\phi_{\xi,\mu}^3} \tilde{w} \\
& \mathcal{J}_{\xi,\mu} \eta \\
&= -\frac{\dot{\psi}_{\xi,\mu}^3}{\phi_{\xi,\mu}^3} \frac{\partial^2 \eta}{(\partial x_0)^2} - \frac{1}{\varepsilon} \left( \frac{\dot{\psi}_{\xi,\mu} \ddot{\psi}_{\xi,\mu}}{\phi_{\xi,\mu}^3} + 2(\phi_{\xi,\mu}^2 - \tau_{\xi,\mu}) \frac{\dot{\phi}_{\xi,\mu} \dot{\psi}_{\xi,\mu}}{\phi_{\xi,\mu}^4} \right) \frac{\partial \eta}{\partial x_0} \\
& \quad - \phi_{\xi,\mu}^{-2} (2\dot{\phi}_{\xi,\mu} \ddot{\psi}_{\xi,\mu} + 2\frac{\dot{\phi}_{\xi,\mu}^2 \dot{\psi}_{\xi,\mu}}{\phi_{\xi,\mu}} + \frac{\dot{\psi}_{\xi,\mu}^3}{\phi_{\xi,\mu}} \\
& \quad - 2\frac{\dot{\psi}_{\xi,\mu}^2}{\phi_{\xi,\mu}} (\phi_{\xi,\mu}^2 - \tau_{\xi,\mu}) - 2\phi_{\xi,\mu} \dot{\phi}_{\xi,\mu}^2) R(\eta, X_0) X_0 \\
& T_{\xi,\mu}(\tilde{w}, \eta) \\
&= \varepsilon L_{\xi,\mu}^1(\tilde{w}, \eta) + \varepsilon^{-1} Q_{\xi,\mu}^1(\tilde{w}, \eta) + \varepsilon^2 L_{\xi,\mu}^2(\partial^2 \eta) + Q_{\xi,\mu}^2(\tilde{w}, \partial^2 \eta).
\end{aligned}$$

**Theorem 4.1.** 1. *The operator*

$$\tilde{\mathcal{L}}_{\xi,\mu} : \tilde{\Pi} C_{\varepsilon}^{2,\alpha}(S\mathcal{N}\Gamma) \rightarrow \tilde{\Pi} C_{\varepsilon}^{0,\alpha}(S\mathcal{N}\Gamma)$$

*is invertible and for some uniform  $C$*

$$\|\tilde{w}\|_{C_{\varepsilon}^{2,\alpha}} \leq C \|\tilde{\mathcal{L}}_{\xi,\mu} \tilde{w}\|_{C_{\varepsilon}^{0,\alpha}}.$$

2. *The operator*

$$\mathcal{J}_{\xi,\mu} : C_{x_0,\varepsilon}^{2,\alpha}(N\Gamma) \rightarrow C_{\varepsilon}^{\alpha}(N\Gamma)$$

*is invertible and for some uniform  $C$*

$$\|\eta\|_{C_{x_0,\varepsilon}^{2,\alpha}} \leq C \|\mathcal{J}_{\xi,\mu} \eta\|_{C_{\varepsilon}^{\alpha}}. \quad (42)$$

3. *If*

$$\mathcal{J}_{\xi,\mu} \hat{\eta} = -\frac{2}{3} \varepsilon \dot{\phi}_{\xi,\mu} P_1(R(\Upsilon_{\theta}, X_0, \Upsilon, \Upsilon_{\theta})),$$

*we have*

$$\|\hat{\eta}\|_{C_{x_0,\varepsilon}^{2,\alpha}} \leq C \varepsilon^2.$$

*Proof.* For the first item, consider the bilinear functional on the space  $\tilde{\Pi} W_{\varepsilon}^{1,2}(S\mathcal{N}\Gamma)$

$$B_{\xi,\mu}(w, v) = \int_{S\mathcal{N}\Gamma} (v(\phi_{\xi,\mu} \tilde{\mathcal{L}}_{\xi,\mu}) w) d\theta d\psi.$$

By using exactly the same argument as used in High mode section, we can prove the results.

For the second and third item, consider  $\frac{\dot{\psi}_{\xi,\mu}^3}{\phi_{\xi,\mu}} \mathcal{J}_{\xi,\mu} \eta$ . Let  $\frac{\partial}{\partial y_0} = \frac{\dot{\psi}_{\xi,\mu}^3}{\phi_{\xi,\mu}^2} \frac{\partial}{\partial x_0}$ . From Corollary 3.19, we know  $y_0(L_\Gamma) = I'_1 L_\Gamma$ ,  $I'_1 = I_1 + O(\varepsilon^2)$ .

$$\begin{aligned}
& \frac{\dot{\psi}_{\xi,\mu}^3}{\phi_{\xi,\mu}} \mathcal{J}_{\xi,\mu} \eta \\
&= -\frac{\partial^2 \eta}{\partial y_0^2} \Big|_{y_0} - \Psi_1(\phi_{\xi,\mu}, \frac{\partial \phi_{\xi,\mu}}{\partial \psi}) R(\eta, X_0) X_0|_{x_0} + \frac{1}{\varepsilon} \rho L(\eta) \\
&= \tilde{\mathcal{J}}'_A \eta + (I'_2 - \Psi_1|_{x_0}) R_{I_1'^{-1} y_0}(\eta(x_0), X_0) X_0 \\
&\quad + \Psi_1|_{x_0} (R_{I_1'^{-1} y_0} - R|_{x_0})(\eta(x_0), X_0) X_0 + \frac{1}{\varepsilon} \rho L(\eta) \\
&= \tilde{\mathcal{J}}'_A \eta + (I'_2 - \Psi_1|_{x_0}) R_{I_1'^{-1} y_0}(\eta(x_0), X_0) X_0 + \varepsilon L(\eta),
\end{aligned}$$

where  $I'_2 = I_2 + O(\varepsilon^2)$  and  $I_1'^2 I'_2 = 1$ .  $I_1'^2 \tilde{\mathcal{J}}'_A$  has the same form as  $\mathcal{J}_A$ . The rest of the proof is similar to Theorem 3.2. However, the primitive of  $(I'_2 - \Psi_1|_{x_0})$  and  $\frac{\dot{\psi}_{\xi,\mu}^3}{\phi_{\xi,\mu}} \dot{\phi}_{\xi,\mu}$  may not be global smooth function on the geodesic. Nevertheless, from Corollary 3.12 we know

$$\begin{aligned}
\chi(y_1) &= \int_0^{y_1} (I'_2 - \Psi_1|_{x_0}) dy_0 = O(\varepsilon), \\
\chi(L_\Gamma) &= O(\varepsilon^2).
\end{aligned}$$

We can apply Lemma 3.3 by noticing

$$I'_2 - \Psi_1|_{x_0} = (I'_2 - \Psi_1|_{x_0} - \frac{\chi(L_\Gamma)}{L_\Gamma}) + \frac{\chi(L_\Gamma)}{L_\Gamma}.$$

For  $\frac{\dot{\psi}_{\xi,\mu}^3}{\phi_{\xi,\mu}} \dot{\phi}_{\xi,\mu}$  we can do the same thing. So all the argument in Theorem 3.2 works here and we can prove this theorem.  $\square$

From the second remark after (13) we have

**Lemma 4.2.**

$$\begin{aligned}
& \|L_{\xi,\mu}^i(w_1, \eta_1) - L_{\xi,\mu}^i(w_2, \eta_2)\|_{C_\varepsilon^\alpha} \\
& \leq C \|w_1 - w_2\|_{C_\varepsilon^{2,\alpha}} + C \|\eta_1 - \eta_2\|_{C_{x_0,\varepsilon}^{1,\alpha}}, i = 1, 3,
\end{aligned}$$

$$\begin{aligned}
& \|L_{\xi,\mu}^2(w_1, \partial_{x_0}^2 \eta_1) - L_{\xi,\mu}^2(w_2, \partial_{x_0}^2 \eta_2)\|_{C_\varepsilon^\alpha} \\
& \leq C \|w_1 - w_2\|_{C_\varepsilon^{2,\alpha}} + C \|\eta_1 - \eta_2\|_{C_{x_0,\varepsilon}^{2,\alpha}},
\end{aligned}$$

$$\begin{aligned}
& \|Q_{\xi,\mu}^i(w_1, \eta_1) - Q_{\xi,\mu}^i(w_2, \eta_2)\|_{C_\varepsilon^\alpha} \\
& \leq C (\|\eta_1\|_{C_{x_0,\varepsilon}^{1,\alpha}} + \|\eta_2\|_{C_{x_0,\varepsilon}^{1,\alpha}} + \|w_1\|_{C_\varepsilon^{2,\alpha}} + \|w_2\|_{C_\varepsilon^{2,\alpha}}) \\
& \quad (\|w_1 - w_2\|_{C_\varepsilon^{2,\alpha}} + \|\eta_1 - \eta_2\|_{C_{x_0,\varepsilon}^{1,\alpha}}), i = 1, 3,
\end{aligned}$$



$$\begin{aligned}
& \|Q_{\xi,\mu}^2(w_1, \partial_{x_0}^2 \eta_1) - Q_{\xi,\mu}^2(w_2, \partial_{x_0}^2 \eta_2)\|_{C_\varepsilon^\alpha} \\
& \leq C(\|\eta_1\|_{C_{x_0}^{2,\alpha}} + \|\eta_2\|_{C_{x_0}^{2,\alpha}} + \|w_1\|_{C_\varepsilon^{2,\alpha}} + \|w_2\|_{C_\varepsilon^{2,\alpha}}) \\
& \quad (\|w_1 - w_2\|_{C_\varepsilon^{2,\alpha}} + \|\eta_1 - \eta_2\|_{C_{x_0,\varepsilon}^{1,\alpha}}) \\
& \quad + C(\|\eta_1\|_{C_{x_0,\varepsilon}^{1,\alpha}} + \|\eta_2\|_{C_{x_0,\varepsilon}^{1,\alpha}} + \|w_1\|_{C_\varepsilon^{2,\alpha}} + \|w_2\|_{C_\varepsilon^{2,\alpha}})\|\eta_1 - \eta_2\|_{C_{x_0,\varepsilon}^{2,\alpha}}.
\end{aligned}$$

Consider

$$\begin{cases} \tilde{\Pi}(H(\mathcal{D}_{\phi_{\xi,\mu},p_0,\varepsilon}(\tilde{w}, \eta)) - \frac{2}{\varepsilon}) &= 0, \\ \Pi_1(H(\mathcal{D}_{\phi_{\xi,\mu},p_0,\varepsilon}(\tilde{w}, \eta)) - \frac{2}{\varepsilon}) &= 0. \end{cases}$$

From Theorem 4.1 and Lemma 4.2, we can solve this system and get  $(\tilde{w}_{\xi,\mu}, \eta_{\xi,\mu})$  which satisfies

$$\|\tilde{w}_{\xi,\mu}\|_{C_\varepsilon^{2,\alpha}} + \|\eta_{\xi,\mu}\|_{C_{x_0,\varepsilon}^{2,\alpha}} \leq C\varepsilon^2.$$

Moreover, from Theorem 3.6, Theorem 4.1, Lemma 4.2, we can prove

**Lemma 4.3.**

$$\begin{cases} \|\tilde{w}_{\xi+\Delta\xi,\mu+\Delta\mu} - \tilde{w}_{\xi,\mu}\|_{C_\varepsilon^{2,\alpha}} &\leq C\varepsilon^2(\varepsilon\|\Delta\mu\|_{C_\varepsilon^\alpha} + \|\Delta\xi\|_{C_{x_0}^1}), \\ \|\eta_{\xi+\Delta\xi,\mu+\Delta\mu} - \eta_{\xi,\mu}\|_{C_{x_0,\varepsilon}^{2,\alpha}} &\leq C\varepsilon(\varepsilon\|\Delta\mu\|_{C_\varepsilon^\alpha} + \|\Delta\xi\|_{C_{x_0}^1}). \end{cases}$$

One can calculate

$$\begin{aligned}
& H(\mathcal{D}_{\phi_{\xi,\mu},p_0,\varepsilon}(\tilde{w}_{\xi,\mu}, \eta_{\xi,\mu})) \\
& = \frac{2}{\varepsilon} + F_4(\phi_{\xi,\mu}, \zeta_{\xi,\mu})(\Pi_0(R(\Upsilon, X_0, \eta_{\xi,\mu}, \Upsilon)) + \xi(x_0)) \\
& \quad + \varepsilon^2\mu + \Pi_0(E_{\xi,\mu} + T_{\xi,\mu}(\tilde{w}_{\xi,\mu}, \eta_{\xi,\mu}) + \varepsilon^{-1}\rho L_{\xi,\mu}^3(\tilde{w}_{\xi,\mu}, \eta_{\xi,\mu}) \\
& \quad + \varepsilon^{-1}\rho Q_{\xi,\mu}^3(\tilde{w}_{\xi,\mu}, \eta_{\xi,\mu}) + \varepsilon L_{\xi,\mu}^4(\rho)) + \varepsilon^2\omega_{\xi,\mu}\phi_{\xi,\mu}^{-1}\frac{d\phi_{\xi,\mu}}{d\psi}
\end{aligned}$$

Now we define a map

$$\begin{aligned}
\Omega : C_{x_0}^1 \times C_\varepsilon^\alpha &\rightarrow C_{x_0}^1 \times C_\varepsilon^\alpha \\
(\xi, \mu) &\mapsto (\Omega^1(\xi, \mu), \Omega^2(\xi, \mu))
\end{aligned}$$

where

$$\begin{cases} \Omega^1(\xi, \mu) &= -\Pi_0(R(\Upsilon, X_0, \eta_{\xi,\mu}, \Upsilon)) \\ \Omega^2(\xi, \mu) &= -\varepsilon^{-2}\Pi_0(T_{\xi,\mu}(\tilde{w}_{\xi,\mu}, \eta_{\xi,\mu}) + \varepsilon^{-1}\rho L_{\xi,\mu}^3(\tilde{w}_{\xi,\mu}, \eta_{\xi,\mu}) \\ &\quad + \varepsilon^{-1}\rho Q_{\xi,\mu}^3(\tilde{w}_{\xi,\mu}, \eta_{\xi,\mu}) + E_{\xi,\mu} + \varepsilon L_{\xi,\mu}^4(\rho)). \end{cases} \quad (43)$$

From Lemma 4.3, Lemma 4.2 we can prove

**Lemma 4.4.** *For fixed  $C_1, C_2$  if  $\|\xi\|_{C_{x_0}^1} \leq C_1\varepsilon^2, \|\mu\|_{C_\varepsilon^\alpha} \leq C_2$ , there is  $C > 0$  which does not depend on  $\varepsilon$  such that*

$$\begin{aligned}
& \|(\Omega^1(\xi_1, \mu_1), \Omega^2(\xi_1, \mu_1)) - (\Omega^1(\xi_2, \mu_2), \Omega^2(\xi_2, \mu_2))\|_{\varepsilon,\alpha} \\
& \leq C\varepsilon\|(\xi_1 - \xi_2, \mu_1 - \mu_2)\|_{\varepsilon,\alpha}
\end{aligned} \quad (44)$$

where the  $\|(\cdot, \cdot)\|_{\varepsilon,\alpha}$  norm is defined in Sub-section 2.1.3.

If  $\xi^0 = 0, \mu^0 = 0$ , we can get  $\tilde{w}_{0,0}, \eta_{0,0}$ , whose norms only depend on the curvature terms along the geodesic. So there is  $C_7$  which only depends on the norms of the curvatures along the geodesic, such that

$$\begin{aligned}\|\Omega^1(0,0)\|_{C_{x_0}^1} &\leq C_7\varepsilon^2 \\ \varepsilon\|\Omega^2(0,0)\|_{C_\varepsilon^\alpha} &\leq C_7\varepsilon\end{aligned}$$

and if we assume

$$\begin{aligned}\xi^1 &= \Omega^1(0,0) \\ \mu^1 &= \Omega^2(0,0),\end{aligned}$$

from (44)

$$\begin{aligned}\|\Omega^1(\xi^1, \mu^1) - \xi^1\|_{C_{x_0}^1} &\leq C_7\varepsilon^2 \\ \varepsilon\|\Omega^2(\xi^1, \mu^1) - \mu^1\|_{C_\varepsilon^\alpha} &\leq C_7\varepsilon^2,\end{aligned}$$

where we can use the same constant  $C_7$ . Then

$$\|\Omega(\xi^1, \mu^1) - (\xi^1, \mu^1)\|_{\varepsilon, \alpha} \leq 2C_7\varepsilon^2.$$

Let

$$E(5C_7) = \{(\xi, \mu) : \|(\xi, \mu) - (\xi^1, \mu^1)\|_{\varepsilon, \alpha} \leq 5C_7\varepsilon^2\}$$

We assume  $C_1 = C_2 = 10C_7$ . Let

$$E(C_1, C_2) = \{(\xi, \mu) : \|\xi\|_{C_{x_0}^1} \leq C_1\varepsilon^2, \|\mu\|_{C_\varepsilon^\alpha} \leq C_2\}.$$

It is obvious that

$$E(5C_7) \subset E(C_1, C_2).$$

So if  $(\xi_1, \mu_1), (\xi_2, \mu_2) \in E(5C_7)$ , then

$$\|\Omega(\xi_1, \mu_1) - \Omega(\xi_2, \mu_2)\|_{\varepsilon, \alpha} \leq C\varepsilon\|(\xi_1 - \xi_2, \mu_1 - \mu_2)\|_{\varepsilon, \alpha}.$$

If we choose  $\varepsilon$  such that

$$C\varepsilon \leq \frac{1}{100},$$

then  $\Omega_2$  maps  $E(5C_7)$  into itself. Note that  $E(5C_7)$  is a complete metric space. From fixed point theorem, there is a unique

$$(\hat{\xi}, \hat{\mu}) \in E(5C_7)$$

such that

$$\Omega(\hat{\xi}, \hat{\mu}) = (\hat{\xi}, \hat{\mu}).$$

For this  $(\hat{\xi}, \hat{\mu})$ , we have

$$H(\mathcal{D}_{\phi_{\hat{\xi}, \hat{\mu}}, p_0, \varepsilon}(\tilde{w}_{\hat{\xi}, \hat{\mu}}, \eta_{\hat{\xi}, \hat{\mu}})) = \frac{2}{\varepsilon} + \varepsilon^2 \omega_{\hat{\xi}, \hat{\mu}} \phi_{\hat{\xi}, \hat{\mu}}^{-1} \frac{\partial \phi_{\hat{\xi}, \hat{\mu}}}{\partial \psi}.$$

## 4.2 The energy of the surface

So far what we've got is a global smooth surface  $\mathcal{D}_{\phi_{\hat{\xi}, \hat{\mu}}, p_0, \varepsilon}(\tilde{w}_{\hat{\xi}, \hat{\mu}}, \eta_{\hat{\xi}, \hat{\mu}})$  whose mean curvature is

$$\frac{2}{\varepsilon} + \omega_{\hat{\xi}, \hat{\mu}} \varepsilon^2 \phi_{\hat{\xi}, \hat{\mu}}^{-1} \frac{\partial \phi_{\hat{\xi}, \hat{\mu}}}{\partial \psi}.$$

Note that  $\mathcal{D}_{\phi_{\hat{\xi}, \hat{\mu}}, p_0, \varepsilon}(\tilde{w}_{\hat{\xi}, \hat{\mu}}, \eta_{\hat{\xi}, \hat{\mu}})$  can also be written as  $\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w_0 + \tilde{w}_{\hat{\xi}, \hat{\mu}}, \eta_{\hat{\xi}, \hat{\mu}})$  with

$$\|w_0\|_{C_{\varepsilon}^{2, \alpha}} \leq C\varepsilon.$$

Remember at the beginning of 0th mode, we have chosen a point  $p_0$  as  $\psi = 0$ , i.e.  $\phi(\psi)$  attains its local minimum at  $p_0$ . We call  $p_0$  the starting point. From the analysis above, we find that  $\hat{\xi}, \hat{\mu}$  and  $\omega_{\hat{\xi}, \hat{\mu}}$  only depend on  $p_0$ . The last thing we can do is to move the starting point along the geodesic such that  $\omega_{\hat{\xi}, \hat{\mu}} = 0$ . Regard  $\psi \in (-\varepsilon_0, \varepsilon_0)$  as local coordinate about  $p_0$  and  $\psi(p_0) = 0$ . Now choose the starting point as  $\psi = \delta$  instead of  $\psi = 0$ . We can do all the analysis above and get  $(\hat{\xi}_{\delta}, \hat{\mu}_{\delta}, \omega_{\delta} = \omega_{\hat{\xi}_{\delta}, \hat{\mu}_{\delta}}, \phi(0)_{\delta} = \phi(0)_{\hat{\xi}_{\delta}, \hat{\mu}_{\delta}}, \phi_{\delta} = \phi_{\hat{\xi}_{\delta}, \hat{\mu}_{\delta}}, \tilde{w}_{\delta}, \eta_{\delta})$  which satisfy similar estimates as  $\delta = 0$  case and

$$H(\mathcal{D}_{\phi_{\delta}, \delta, \varepsilon}(\tilde{w}_{\delta}, \eta_{\delta})) = \frac{2}{\varepsilon} + \omega_{\delta} \varepsilon^2 \phi_{\delta}^{-1} \frac{\partial \phi_{\delta}}{\partial \psi}.$$

We make the following notations

$$\begin{aligned} \frac{\partial f_{\delta}(\psi)}{\partial \delta} \Big|_{\delta=0} &= \lim_{\delta \rightarrow 0} \frac{f_{\delta}(\psi) - f_0(\psi)}{\delta} \\ \frac{\partial' f_{\delta}(\psi)}{\partial \delta} \Big|_{\delta=0} &= \lim_{\delta \rightarrow 0} \frac{f_{\delta}(\psi) - f_0(\psi - \delta)}{\delta}. \end{aligned}$$

We want to study  $\frac{\partial' \phi_{\delta}}{\partial \delta}, \frac{\partial' \tilde{w}_{\delta}}{\partial \delta}, \frac{\partial' \eta_{\delta}}{\partial \delta}$ . We can take a new point of view. Identify the starting points and translate the curvature terms along the geodesic. We know

$$\frac{\partial'}{\partial \delta} R_i = \varepsilon \bar{R}_i \tag{45}$$

and  $\bar{R}_i$ 's derivatives with respect to  $x_0$  are bounded. Besides,  $\bar{R}_i$  belongs to the same subspace (range of  $\Pi_0, \Pi_1, \tilde{\Pi}$ ) as  $R_i$  does.

Now we start with the initial surface  $\mathcal{D}_{\phi_{\hat{\xi}, \hat{\mu}}, p_0, \varepsilon}(\tilde{w}_{\hat{\xi}, \hat{\mu}}, \eta_{\hat{\xi}, \hat{\mu}})$ .  $\phi_{\hat{\xi}, \hat{\mu}}$  satisfies (20) with  $\xi = \hat{\xi}, \mu = \hat{\mu}, \phi(0) = \phi(0)_{\hat{\xi}, \hat{\mu}}, \omega = \omega_{\hat{\xi}, \hat{\mu}}$ . Heuristically when  $\delta$  has a variation of size 1, the  $C_{x_0}^1$  norm of  $R_i$  will have a variation of  $O(\varepsilon)$ . At first we fix  $\xi = \hat{\xi}$  and  $\mu = \hat{\mu}$ . By analyzing both nonlinear ODE and its linearizations, we find  $\omega$  should be perturbed as large as  $O(\varepsilon)$  and  $\phi(0)$  should be perturbed as large as  $O(\varepsilon^3)$  to match the boundary value. Hence we know  $\|\phi_{\delta}(\psi + \delta) - \phi_{\hat{\xi}, \hat{\mu}}(\psi)\|_{C_{\varepsilon}^{2, \alpha}} = O(\varepsilon^2)$ . From (41), Theorem 4.1 and Lemma 4.2 we have  $\|\tilde{w}_{\delta}(\psi + \delta) - \tilde{w}(\psi)\|_{C_{\varepsilon}^{2, \alpha}} \leq C\varepsilon^3, \|\eta_{\delta}(\psi + \delta) - \eta(\psi)\|_{C_{x_0, \varepsilon}^{2, \alpha}} \leq C\varepsilon^3$ . Now we replace  $(\hat{\xi}, \hat{\mu})$  by  $(\Omega^1(\hat{\xi}, \hat{\mu}), \Omega^2(\hat{\xi}, \hat{\mu}))$ . From (43),  $\hat{\xi}$  has a  $O(\varepsilon^3)$  variation and  $\hat{\mu}$  has a variation of  $O(\varepsilon)$  which implies an  $\varepsilon^2$  variation again on  $\phi_{\delta}$  by (23) and hence a  $O(\varepsilon^4)$  variation on  $\tilde{w}_{\delta}$  and a

$O(\varepsilon^3)$  variation on  $\eta_\delta$ . However, this time  $\hat{\mu}$  only has a variation of  $O(\varepsilon^2)$ . So the iteration argument works.

	1	2	3	...	k	...
$\omega_\delta$ $ \cdot $	$\varepsilon$	$\varepsilon$	$\varepsilon^2$	...	$\varepsilon^{k-1}$	...
$\phi_\delta(\delta)$ $ \cdot $	$\varepsilon^3$	$\varepsilon^3$	$\varepsilon^4$	...	$\varepsilon^{k+1}$	...
$\phi_\delta(\cdot + \delta)$ $\ \cdot\ _{C_\varepsilon^{2,\alpha}}$	$\varepsilon^2$	$\varepsilon^2$	$\varepsilon^3$	...	$\varepsilon^k$	...
$\tilde{w}_\delta(\cdot + \delta)$ $\ \cdot\ _{C_\varepsilon^{2,\alpha}}$	$\varepsilon^3$	$\varepsilon^4$	$\varepsilon^5$	...	$\varepsilon^{k+2}$	...
$\eta_\delta(\cdot + \delta)$ $\ \cdot\ _{C_{x_0,\varepsilon}^{2,\alpha}}$	$\varepsilon^3$	$\varepsilon^3$	$\varepsilon^4$	...	$\varepsilon^{k+1}$	...
$\xi_\delta(\cdot + \delta)$ $\ \cdot\ _{C_{x_0}^1}$	$\varepsilon^3$	$\varepsilon^3$	$\varepsilon^4$	...	$\varepsilon^{k+1}$	...
$\mu_\delta(\cdot + \delta)$ $\ \cdot\ _{C_\varepsilon^\alpha}$	$\varepsilon$	$\varepsilon^2$	$\varepsilon^3$	...	$\varepsilon^k$	...

In the above form, for example, the  $\varepsilon^{k+1}$  in  $(\eta_\delta(\cdot + \delta), k)$  position means in the  $k$ th step of iteration,  $\eta_\delta(\cdot + \delta)$  has a variation of  $\varepsilon^{k+1}$  measured in  $\|\cdot\|_{C_{x_0,\varepsilon}^{2,\alpha}}$ .

The above argument can be made precise by taking  $\frac{\partial'}{\partial\delta}$  derivative to each mode as well as the expression (43).

At last we get

$$\begin{aligned} \left\| \frac{\partial'}{\partial\delta} \phi_\delta|_{\delta=0} \right\|_{C_\varepsilon^{2,\alpha}} &\leq C\varepsilon^2, \\ \left\| \frac{\partial'}{\partial\delta} \eta_\delta|_{\delta=0} \right\|_{C_{x_0,\varepsilon}^{2,\alpha}} &\leq C\varepsilon^3, \\ \left\| \frac{\partial'}{\partial\delta} \tilde{w}_\delta|_{\delta=0} \right\|_{C_\varepsilon^{2,\alpha}} &\leq C\varepsilon^3. \end{aligned}$$

So we have

$$\begin{aligned} &\frac{\partial}{\partial\delta}(\varepsilon(\phi_\delta + \tilde{w}_\delta) + \langle \eta_\delta, \Upsilon \rangle)|_{\delta=0} \\ &= \frac{\partial'}{\partial\delta}(\varepsilon(\phi_\delta + \tilde{w}_\delta) + \langle \eta_\delta, \Upsilon \rangle)|_{\delta=0} - \frac{\partial}{\partial\psi}(\varepsilon(\phi_{\hat{\xi},\hat{\mu}} + \tilde{w}_{\hat{\xi},\hat{\mu}}) + \langle \eta_{\hat{\xi},\hat{\mu}}, \Upsilon \rangle)|_{\psi=0} \\ &= -\varepsilon \frac{\partial\phi_{\hat{\xi},\hat{\mu}}}{\partial\psi} + O(\varepsilon^3). \end{aligned}$$

Consider the energy functional of the surface  $\Sigma_\delta = \mathcal{D}_{\phi_\delta, \delta, \varepsilon}(\tilde{w}_\delta, \eta_\delta)$

$$E(\Sigma_\delta) = \text{Area}(\Sigma_\delta) - \frac{2}{\varepsilon} \text{Vol}(\Sigma_\delta).$$

We have

$$\begin{aligned}
& \frac{d}{d\delta} E(\Sigma_\delta)|_{\delta=0} \\
&= \frac{d}{d\delta} (\text{Area}(\Sigma_\delta) - \frac{2}{\varepsilon} \text{Vol}(\Sigma_\delta)) \\
&= \int_{\Sigma_0} H \frac{\partial}{\partial \delta} (\varepsilon(\phi_\delta(\psi) + \tilde{w}_\delta) + \langle \eta_\delta, \Upsilon \rangle)|_{\delta=0} < N, \Upsilon > dS \\
&\quad - \frac{2}{\varepsilon} \int_{\Sigma_0} \frac{\partial}{\partial \delta} (\varepsilon(\phi_\delta(\psi) + \tilde{w}_\delta) + \langle \eta_\delta, \Upsilon \rangle)|_{\delta=0} < N, \Upsilon > dS \\
&= -\varepsilon^3 \omega_{\hat{\xi}, \hat{\mu}} \int_{\Sigma_0} \phi_{\hat{\xi}, \hat{\mu}}^{-1} \left[ \left( \frac{\partial \phi_{\hat{\xi}, \hat{\mu}}}{\partial \psi} \right)^2 + O(\varepsilon^2) \right] < N, \Upsilon > dS,
\end{aligned}$$

where  $\Sigma_0 = \mathcal{D}_{\phi_{\hat{\xi}, \hat{\mu}}, p_0, \varepsilon}(\tilde{w}_{\hat{\xi}, \hat{\mu}}, \eta_{\hat{\xi}, \hat{\mu}})$ . Note that

$$\int_{\Sigma_0} \phi_{\hat{\xi}, \hat{\mu}}^{-1} \left( \frac{\partial \phi_{\hat{\xi}, \hat{\mu}}}{\partial \psi} \right)^2 < N, \Upsilon > dS$$

is always positive.

Similarly for  $\delta \in [0, \frac{L_\Gamma}{\varepsilon}]$ , we can prove that

$$\begin{aligned}
& \frac{d}{d\delta} E(\Sigma_\delta)|_{\delta=0} \\
&= -\varepsilon^3 \omega_\delta \int_{\Sigma_\delta} \phi_\delta^{-1} \left[ \left( \frac{\partial \phi_\delta}{\partial \psi} \right)^2 + O(\varepsilon^2) \right] < N, \Upsilon > dS
\end{aligned}$$

with the integral being always positive. If  $E$  is constant when  $\delta \in [0, \frac{L_\Gamma}{\varepsilon}]$ , we have for every  $\delta$ ,  $\omega_\delta = 0$ . Then we have infinitely many Delaunay type constant mean curvature surfaces. If  $E$  is not always constant, we will at least get two zeros of  $\frac{d}{d\delta} E(\Sigma_\delta)$ , where we have  $\omega_\delta = 0$ . Then we get two Delaunay type constant mean curvature surfaces. The two surfaces are not the same, because they correspond to the maximal value and minimal value of  $E$ . That the Delaunay type CMC surfaces are embedded is evident. First the CMC surfaces constructed can be regarded as a smooth map from  $T^2$  to  $M$ , because the unit normal bundle of  $\Gamma$  has  $T^2$  topology. And because  $T^2$  is compact topological space, we need only to prove that this map is injective. This is easily seen from the fact that  $\Gamma$  is embedded and the estimates of the functions in each mode. So we proved the main theorem.

For Corollary 1.2, we see that the non-degeneracy condition of the Jacobi operator of the geodesic is only used in 1st mode. However, when the metric around  $\Gamma$  has rotational symmetry, i.e.  $\frac{\partial}{\partial \theta}$  is killing vector field, if we look at the expression of mean curvature (13), we will find that  $R_1, R_2$  and  $E$  have no 1st mode or high mode projections. So we may assume  $\eta = 0$  and  $\tilde{w} = 0$ . The only thing that need to do is to solve the 0th mode. So we can use the procedure in Sub-section 3.4 to solve 0th mode up to the kernel  $\frac{\partial \phi}{\partial \psi}$ . And then we use the same argument as in this Sub-section to remove the kernel. So we can get the CMC surfaces of Delaunay type.

## A The calculation of the mean curvature

We prove (13) here. By definition we have

$$\begin{aligned}
H(\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)) &= g^{ss} \langle N, \nabla_{\partial_s} \partial_s \rangle + 2g^{s\theta} \langle N, \nabla_{\partial_\theta} \partial_s \rangle + g^{\theta\theta} \langle N, \nabla_{\partial_\theta} \partial_\theta \rangle \\
&= \frac{1}{k} (g^{ss} \langle kN, \nabla_{\partial_s} \partial_s \rangle + 2g^{s\theta} \langle kN, \nabla_{\partial_\theta} \partial_s \rangle \\
&\quad + g^{\theta\theta} \langle kN, \nabla_{\partial_\theta} \partial_\theta \rangle).
\end{aligned}$$

For this we have to calculate the second fundamental form.

$$g^{ss} \langle kN, \nabla_{\partial_s} \partial_s \rangle = g^{ss} \langle N_0 + a_1 \partial_s + a_2 \partial_\theta, \nabla_{\partial_s} \partial_s \rangle$$

$$\begin{aligned}
\langle \partial_s, \nabla_{\partial_s} \partial_s \rangle &= \frac{1}{2} \partial_s \langle \partial_s, \partial_s \rangle \\
&= \frac{\varepsilon^2}{2} \partial_s (\phi^2 + \varepsilon^2 \phi^2 \dot{\psi}^2 R(\Upsilon, X_0, \Upsilon, X_0) + 2\varepsilon \phi \dot{\psi}^2 R(\Upsilon, X_0, \eta, X_0) \\
&\quad + \frac{4}{3} \varepsilon \phi \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \eta, \Upsilon) + 2\dot{\phi} \frac{\partial w}{\partial s} + 2\dot{\phi} \dot{\psi} \langle \Upsilon, \frac{\partial \eta}{\partial x_0} \rangle_e \\
&\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)), \\
&= \frac{\varepsilon^2}{2} (2\phi \dot{\phi} + L(w, \eta) + \varepsilon L(\frac{\partial^2 \eta}{\partial x_0^2}) + Q(w, \eta) + O(\varepsilon^2)), \tag{46}
\end{aligned}$$

$$\begin{aligned}
\langle \partial_\theta, \nabla_{\partial_s} \partial_s \rangle &= \partial_s \langle \partial_\theta, \partial_s \rangle - \langle \nabla_{\partial_s} \partial_\theta, \partial_s \rangle \\
&= \partial_s \langle \partial_\theta, \partial_s \rangle - \frac{1}{2} \partial_\theta \langle \partial_s, \partial_s \rangle \\
&= \varepsilon^2 \partial_s (\frac{2}{3} \varepsilon^2 \phi^3 \dot{\psi} R(\Upsilon, X_0, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon \phi^2 \dot{\psi} (R(\Upsilon, X_0, \eta, \Upsilon_\theta) \\
&\quad + R(\eta, X_0, \Upsilon, \Upsilon_\theta)) + \frac{1}{3} \varepsilon \phi^2 \dot{\phi} R(\eta, \Upsilon, \Upsilon, \Upsilon_\theta) + \dot{\phi} \frac{\partial w}{\partial \theta} \\
&\quad + \phi \dot{\psi} \langle \frac{\partial \eta}{\partial x_0}, \Upsilon_\theta \rangle_e + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \\
&\quad - \frac{\varepsilon^2}{2} \partial_\theta (\phi^2 + \varepsilon^2 \phi^2 \dot{\psi}^2 R(\Upsilon, X_0, \Upsilon, X_0) + 2\varepsilon \phi \dot{\psi}^2 R(\Upsilon, X_0, \eta, X_0) \\
&\quad + \frac{4}{3} \varepsilon \phi \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \eta, \Upsilon) + 2\dot{\phi} \frac{\partial w}{\partial s} + 2\dot{\phi} \dot{\psi} \langle \Upsilon, \frac{\partial \eta}{\partial x_0} \rangle_e \\
&\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)), \\
&= \varepsilon^2 (O(\varepsilon^2) + L(w, \eta) + \varepsilon L(\frac{\partial^2 \eta}{\partial x_0^2}) + Q(w, \eta)) \tag{47}
\end{aligned}$$

$$\begin{aligned}
\nabla_{\partial_s} \partial_s &= \varepsilon \nabla_{\partial_s} (\dot{\psi} X_0 + (\dot{\phi} + \frac{\partial w}{\partial s}) \Upsilon + \dot{\psi} \frac{\partial \eta^i}{\partial x_0} X_i) \\
&= \varepsilon (\ddot{\psi} X_0 + (\ddot{\phi} + \frac{\partial^2 w}{\partial s^2}) \Upsilon + (\ddot{\psi} \frac{\partial \eta^i}{\partial x_0} + \dot{\psi}^2 \varepsilon \frac{\partial^2 \eta^i}{\partial x_0^2}) X_i) \\
&\quad + \varepsilon^2 (\dot{\psi}^2 \nabla_{X_0} X_0 + (\dot{\phi} + \frac{\partial w}{\partial s})^2 \nabla_{\Upsilon} \Upsilon + \dot{\psi}^2 (\frac{\partial \eta^i}{\partial x_0}) (\frac{\partial \eta^j}{\partial x_0}) \nabla_{X_i} X_j) \\
&\quad + 2 \dot{\psi} (\dot{\phi} + \frac{\partial w}{\partial s}) \nabla_{X_0} \Upsilon + 2 \dot{\psi}^2 \frac{\partial \eta^i}{\partial x_0} \nabla_{X_0} X_i + 2 (\dot{\phi} + \frac{\partial w}{\partial s}) \dot{\psi} \frac{\partial \eta^i}{\partial x_0} \nabla_{\Upsilon} X_i)
\end{aligned}$$

We calculate

$$\langle N_0, \nabla_{\partial_s} \partial_s \rangle = \langle \frac{1}{\phi} (\dot{\phi} X_0 - \dot{\psi} \Upsilon), \nabla_{\partial_s} \partial_s \rangle$$

term by term. There would be 20 terms totally.

$$\begin{aligned}
\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon \ddot{\psi} X_0 \rangle &= \varepsilon \frac{\dot{\phi} \ddot{\psi}}{\phi} \langle X_0, X_0 \rangle \\
&= \varepsilon \frac{\dot{\phi} \ddot{\psi}}{\phi} (1 + \varepsilon^2 \phi^2 R(\Upsilon, X_0, \Upsilon, X_0)_p + 2\varepsilon \phi R(\Upsilon, X_0, \eta, X_0) \\
&\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \\
&= \varepsilon \frac{\dot{\phi} \ddot{\psi}}{\phi} + \varepsilon^3 \phi \dot{\phi} \ddot{\psi} R(\Upsilon, X_0, \Upsilon, X_0) + 2\varepsilon^2 \dot{\phi} \ddot{\psi} R(\Upsilon, X_0, \eta, X_0) \\
&\quad + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4),
\end{aligned}$$

$$\begin{aligned}
\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon (\ddot{\phi} + \frac{\partial^2 w}{\partial s^2}) \Upsilon \rangle &= \varepsilon \frac{\dot{\phi}}{\phi} (\ddot{\phi} + \frac{\partial^2 w}{\partial s^2}) \langle X_0, \Upsilon \rangle \\
&= \varepsilon \frac{\dot{\phi}}{\phi} (\ddot{\phi} + \frac{\partial^2 w}{\partial s^2}) (\frac{2}{3} \varepsilon \phi R(\Upsilon, X_0, \eta, \Upsilon) \\
&\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \\
&= \frac{2}{3} \varepsilon^2 \dot{\phi} \ddot{\phi} R(\Upsilon, X_0, \eta, \Upsilon) + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4),
\end{aligned}$$

$$\begin{aligned}
\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon \ddot{\psi} \frac{\partial \eta^i}{\partial x_0} X_i \rangle &= \varepsilon \frac{\dot{\phi} \ddot{\psi}}{\phi} \frac{\partial \eta^i}{\partial x_0} \langle X_0, X_i \rangle \\
&= \varepsilon \frac{\dot{\phi} \ddot{\psi}}{\phi} \frac{\partial \eta^i}{\partial x_0} (O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta) + O(\varepsilon)) \\
&= \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta),
\end{aligned}$$

$$\begin{aligned}
\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 \dot{\psi}^2 \frac{\partial^2 \eta^i}{\partial x_0^2} X_i \rangle &= \varepsilon^2 \frac{\dot{\phi} \dot{\psi}^2}{\phi} \frac{\partial^2 \eta^i}{\partial x_0^2} (O(\varepsilon^2) + \varepsilon L(w, \eta) \\
&\quad + Q(w, \eta) + O(\varepsilon)) \\
&= \varepsilon^4 L(\partial_{x_0}^2 \eta) + \varepsilon^2 Q(w, \partial_{x_0}^2 \eta),
\end{aligned} \tag{49}$$

$$\begin{aligned}
\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 \dot{\psi}^2 \nabla_{X_0} X_0 \rangle &= \varepsilon^2 \frac{\dot{\phi} \dot{\psi}^2}{\phi} \langle X_0, \nabla_{X_0} X_0 \rangle \\
&= \varepsilon^2 \frac{\dot{\phi} \dot{\psi}^2}{\phi} (O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta)) \\
&= O(\varepsilon^4) + \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta),
\end{aligned}$$

$$\begin{aligned}
\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\dot{\phi} + \frac{\partial w}{\partial s})^2 \nabla_{\Upsilon} \Upsilon \rangle &= \varepsilon^2 \frac{\dot{\phi}}{\phi} (\dot{\phi} + \frac{\partial w}{\partial s})^2 \langle X_0, \nabla_{\Upsilon} \Upsilon \rangle \\
&= \varepsilon^2 \frac{\dot{\phi}}{\phi} (\dot{\phi} + \frac{\partial w}{\partial s})^2 (\Upsilon \langle X_0, \Upsilon \rangle - \langle \nabla_{X_0} \Upsilon, \Upsilon \rangle).
\end{aligned}$$

The method to calculate this will be used several times, so we write it in detail. Although  $\Upsilon$  actually depends on  $\theta$ , here we are only interested in  $\nabla_{\Upsilon} \Upsilon$ , we may pretend  $\Upsilon$  is constant vector in the coordinates  $\{x_0, x_1, x_2\}$ . We may make such assumption where it is convenient. Evidently  $\Upsilon(x_k) = \Upsilon^k$ . So

$$\begin{aligned}
\Upsilon \langle X_0, \Upsilon \rangle &= \Upsilon \left( \frac{2}{3} R(X_k, X_0, X_l, \Upsilon)_p x_k x_l + O(r^3) \right) \\
&= \frac{2}{3} R(X_k, X_0, X_l, \Upsilon)_p (\Upsilon^k (\varepsilon(\phi + w) \Upsilon^l + \eta^l) \\
&\quad + (\varepsilon(\phi + w) \Upsilon^k + \eta^k) \Upsilon^l) + O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta) \\
&= \frac{2}{3} R(\Upsilon, X_0, \eta, \Upsilon) + O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta),
\end{aligned}$$

$$\langle \nabla_{X_0} \Upsilon, \Upsilon \rangle_p = 0$$

for all  $p \in \Gamma$ . We consider

$$X_j \langle \nabla_{X_0} \Upsilon, \Upsilon \rangle_p = \langle \nabla_{X_j} \nabla_{X_0} \Upsilon, \Upsilon \rangle_p + \langle \nabla_{X_0} \Upsilon, \nabla_{X_j} \Upsilon \rangle_p.$$

From Lemma 2.1 we know  $(\nabla_{X_\alpha} X_\beta)_p = 0$ . So

$$\begin{aligned}
X_j \langle \nabla_{X_0} \Upsilon, \Upsilon \rangle_p &= \langle \nabla_{X_j} \nabla_{X_0} \Upsilon, \Upsilon \rangle_p \\
&= \langle \nabla_{X_0} \nabla_{X_j} \Upsilon, \Upsilon \rangle_p + R(X_j, X_0, \Upsilon, \Upsilon)_p.
\end{aligned}$$

We know  $R(X_j, X_0, \Upsilon, \Upsilon)_p = 0$ . And we know  $\nabla_{X_j} \Upsilon = 0$  always holds on the geodesic. So  $(\nabla_{X_0} \nabla_{X_j} \Upsilon)_p = 0$ . So  $X_j \langle \nabla_{X_0} \Upsilon, \Upsilon \rangle_p = 0$ . So we know

$$\langle \nabla_{X_0} \Upsilon, \Upsilon \rangle(x_0, x_1, x_2) = O(r^2) = O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta).$$



Now we get

$$\begin{aligned} \langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\dot{\phi} + \frac{\partial w}{\partial s})^2 \nabla_{\Upsilon} \Upsilon \rangle &= \frac{2}{3} \varepsilon^2 \frac{\dot{\phi}^3}{\phi} R(\Upsilon, X_0, \eta, \Upsilon) \\ &\quad + O(\varepsilon^4) + \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta), \end{aligned}$$

By using the skill above we can calculate all the remaining terms. We state the result directly.

$$\varepsilon^2 \langle \frac{\dot{\phi}}{\phi} X_0, \dot{\psi}^2 (\frac{\partial \eta^i}{\partial x_0}) (\frac{\partial \eta^j}{\partial x_0}) \nabla_{X_i} X_j \rangle = \varepsilon^3 Q(w, \eta),$$

$$\begin{aligned} \varepsilon^2 \langle \frac{\dot{\phi}}{\phi} X_0, 2\dot{\psi} (\dot{\phi} + \frac{\partial w}{\partial s}) \nabla_{X_0} \Upsilon \rangle &= 2\varepsilon^3 \dot{\phi}^2 \dot{\psi} R(\Upsilon, X_0, \Upsilon, X_0) \\ &\quad + 2\varepsilon^2 \frac{\dot{\phi}^2 \dot{\psi}}{\phi} R(\Upsilon, X_0, \eta, X_0) \\ &\quad + O(\varepsilon^4) + \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta), \end{aligned}$$

$$\varepsilon^2 \langle \frac{\dot{\phi}}{\phi} X_0, 2\dot{\psi}^2 \frac{\partial \eta^i}{\partial x_0} \nabla_{X_0} X_i \rangle = \varepsilon^3 L(w, \eta),$$

$$\varepsilon^2 \langle \frac{\dot{\phi}}{\phi} X_0, 2(\dot{\phi} + \frac{\partial w}{\partial s}) \dot{\psi} \frac{\partial \eta^i}{\partial x_0} \nabla_{\Upsilon} X_i \rangle = \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta),$$

$$\begin{aligned} \langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon \ddot{\psi} X_0 \rangle &= -\varepsilon \frac{\dot{\psi} \ddot{\psi}}{\phi} \langle \Upsilon, X_0 \rangle \\ &= -\frac{2}{3} \varepsilon^2 \dot{\psi} \ddot{\psi} R(\Upsilon, X_0, \eta, \Upsilon) + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4), \end{aligned}$$

$$\begin{aligned} \langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon (\ddot{\phi} + \frac{\partial^2 w}{\partial s^2}) \Upsilon \rangle &= -\varepsilon \frac{\ddot{\phi} \dot{\psi}}{\phi} - \varepsilon \frac{\dot{\psi}}{\phi} \frac{\partial^2 w}{\partial s^2} \\ &\quad + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4), \end{aligned}$$

$$\langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon \ddot{\psi} \frac{\partial \eta^i}{\partial x_0} X_i \rangle = -\varepsilon \frac{\dot{\psi} \ddot{\psi}}{\phi} \langle \frac{\partial \eta}{\partial x_0}, \Upsilon \rangle,$$

$$\langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon^2 \dot{\psi}^2 \frac{\partial^2 \eta^i}{\partial x_0^2} X_i \rangle = -\varepsilon^2 \frac{\dot{\psi}^3}{\phi} \langle \frac{\partial^2 \eta}{\partial x_0^2}, \Upsilon \rangle,$$

$$\begin{aligned} \langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon^2 \dot{\psi}^2 \nabla_{X_0} X_0 \rangle &= \varepsilon^3 \dot{\psi}^3 R(\Upsilon, X_0, \Upsilon, X_0) + \varepsilon^2 \frac{\dot{\psi}^3}{\phi} R(\Upsilon, X_0, \eta, X_0) \\ &\quad + \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta) + O(\varepsilon^4), \end{aligned}$$

$$\begin{aligned}
\langle -\frac{\dot{\psi}}{\phi}\Upsilon, \varepsilon^2(\dot{\phi} + \frac{\partial w}{\partial s})^2 \nabla_{\Upsilon} \Upsilon \rangle &= \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta) + O(\varepsilon^4), \\
\langle -\frac{\dot{\psi}}{\phi}\Upsilon, \varepsilon^2 \dot{\psi}^2 (\frac{\partial \eta^i}{\partial x_0}) (\frac{\partial \eta^j}{\partial x_0}) \nabla_{X_i} X_j \rangle &= \varepsilon^3 Q(w, \eta), \\
\langle -\frac{\dot{\psi}}{\phi}\Upsilon, \varepsilon^2 2\dot{\psi}(\dot{\phi} + \frac{\partial w}{\partial s}) \nabla_{X_0} \Upsilon \rangle &= \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta) + O(\varepsilon^4), \\
\langle -\frac{\dot{\psi}}{\phi}\Upsilon, \varepsilon^2 2\dot{\psi}^2 \frac{\partial \eta^i}{\partial x_0} \nabla_{X_0} X_i \rangle &= \varepsilon^3 L(w, \eta), \\
\langle -\frac{\dot{\psi}}{\phi}\Upsilon, \varepsilon^2 2(\dot{\phi} + \frac{\partial w}{\partial s}) \dot{\psi} \frac{\partial \eta^i}{\partial x_0} \nabla_{\Upsilon} X_i \rangle &= \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta).
\end{aligned}$$

Collecting all the terms above and notice that

$$\frac{\dot{\phi}\ddot{\psi}}{\phi} - \frac{\ddot{\phi}\dot{\psi}}{\phi} = \phi^2 - \tau_0 \quad (50)$$

we get

$$\begin{aligned}
\langle N_0, \nabla_{\partial_s} \partial_s \rangle &= \varepsilon(\phi^2 - \tau_0) - \varepsilon \frac{\dot{\psi}}{\phi} \frac{\partial^2 w}{\partial s^2} - \varepsilon^2 \frac{\dot{\psi}^3}{\phi} \langle \frac{\partial^2 \eta}{(\partial x^0)^2}, \Upsilon \rangle - \varepsilon \frac{\dot{\psi}\ddot{\psi}}{\phi} \langle \frac{\partial \eta}{\partial x^0}, \Upsilon \rangle \\
&+ \varepsilon^3 (\phi \dot{\phi} \ddot{\psi} + 2\dot{\phi}^2 \dot{\psi} + \dot{\psi}^3) R(\Upsilon, X_0, \Upsilon, X_0) \\
&+ \varepsilon^2 (2\dot{\phi} \ddot{\psi} + 2\frac{\dot{\phi}^2 \dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi}) R(\Upsilon, X_0, \eta, X_0) \\
&+ \varepsilon^2 (\frac{2}{3} \dot{\phi} \ddot{\phi} + \frac{2}{3} \frac{\dot{\phi}^3}{\phi} - \frac{2}{3} \dot{\psi} \ddot{\psi}) R(\Upsilon, X_0, \eta, \Upsilon) \\
&+ \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + \varepsilon^4 L(\partial^2 \eta) + \varepsilon^2 Q(w, \partial^2 \eta) + O(\varepsilon^4), \quad (51)
\end{aligned}$$

$$g^{\theta\theta} \langle kN, \nabla_{\partial_\theta} \partial_\theta \rangle = g^{\theta\theta} \langle N_0 + a_1 \partial_s + a_2 \partial_\theta, \nabla_{\partial_\theta} \partial_\theta \rangle.$$

$$\begin{aligned}
\langle \partial_s, \nabla_{\partial_\theta} \partial_\theta \rangle &= \partial_\theta \langle \partial_s, \partial_\theta \rangle - \frac{1}{2} \partial_s \langle \partial_\theta, \partial_\theta \rangle \\
&= \varepsilon^2 \partial_\theta (\frac{2}{3} \varepsilon^2 \phi^3 \dot{\psi} R(\Upsilon, X_0, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon \phi^2 \dot{\psi} (R(\Upsilon, X_0, \eta, \Upsilon_\theta) \\
&+ R(\eta, X_0, \Upsilon, \Upsilon_\theta))) + \frac{1}{3} \varepsilon \phi^2 \dot{\phi} R(\eta, \Upsilon, \Upsilon, \Upsilon_\theta) \\
&+ \dot{\phi} \frac{\partial w}{\partial \theta} + \phi \dot{\psi} \langle \frac{\partial \eta}{\partial x_0}, \Upsilon_\theta \rangle + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \\
&- \frac{\varepsilon^2}{2} \partial_s (\phi^2 + 2\phi w + \frac{1}{3} \varepsilon^2 \phi^4 R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) \\
&+ \frac{2}{3} \varepsilon \phi^3 R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \\
&= -\varepsilon^2 \phi \dot{\phi} + O(\varepsilon^4) + \varepsilon^2 L(w, \eta) + \varepsilon^2 Q(w, \eta), \quad (52)
\end{aligned}$$

$$\begin{aligned}
\langle \partial_\theta, \nabla_{\partial_\theta} \partial_\theta \rangle &= \frac{1}{2} \partial_\theta \langle \partial_\theta, \partial_\theta \rangle \\
&= \frac{\varepsilon^2}{2} \partial_\theta (\phi^2 + 2\phi w + \frac{1}{3} \varepsilon^2 \phi^4 R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) \\
&\quad + \frac{2}{3} \varepsilon \phi^3 R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3)) \\
&= O(\varepsilon^4) + \varepsilon^2 L(w, \eta) + \varepsilon^2 Q(w, \eta), \tag{53}
\end{aligned}$$

$$\begin{aligned}
\nabla_{\partial_\theta} \partial_\theta &= \varepsilon \frac{\partial w}{\partial \theta} \Upsilon_\theta + \varepsilon \frac{\partial^2 w}{\partial \theta^2} \Upsilon + \varepsilon^2 (\phi + w)^2 \nabla_{\Upsilon_\theta} \Upsilon_\theta \\
&\quad + \varepsilon^2 \left( \frac{\partial w}{\partial \theta} \right)^2 \nabla_{\Upsilon} \Upsilon + \varepsilon^2 (\phi + w) \frac{\partial w}{\partial \theta} (\nabla_{\Upsilon_\theta} \Upsilon + \nabla_{\Upsilon} \Upsilon_\theta)
\end{aligned}$$

$$\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon \frac{\partial w}{\partial \theta} \Upsilon_\theta \rangle = \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta),$$

$$\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon \frac{\partial^2 w}{\partial \theta^2} \Upsilon \rangle = \varepsilon^4 L(w, \eta) + \varepsilon Q(w, \eta),$$

$$\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\phi + w)^2 \nabla_{\Upsilon_\theta} \Upsilon_\theta \rangle = \varepsilon^2 \frac{\dot{\phi}}{\phi} (\phi + w)^2 \langle X_0, \nabla_{\Upsilon_\theta} \Upsilon_\theta \rangle,$$

notice that

$$\langle X_0, \nabla_{\Upsilon_\theta} \Upsilon_\theta \rangle = \Upsilon_\theta \langle X_0, \Upsilon_\theta \rangle - \langle \nabla_{\Upsilon_\theta} X_0, \Upsilon_\theta \rangle,$$

We recall  $\Upsilon(x_0, x_1, x_2) = \frac{(x_1 - \eta_1, x_2 - \eta_2)}{\sqrt{(x_1 - \eta_1)^2 + (x_2 - \eta_2)^2}} = (\cos \theta, \sin \theta)$  and  $\Upsilon_\theta(x_0, x_1, x_2) = \frac{(-x_2 + \eta_2, x_1 - \eta_1)}{\sqrt{(x_1 - \eta_1)^2 + (x_2 - \eta_2)^2}} = (-\sin \theta, \cos \theta)$ . We denote  $(-x_2 + \eta_2, x_1 - \eta_1)$  by  $\tilde{\partial}_\theta$ , and  $\sqrt{(x_1 - \eta_1)^2 + (x_2 - \eta_2)^2}$  by  $\tilde{r}$

$$\begin{aligned}
\Upsilon_\theta \langle X_0, \Upsilon_\theta \rangle &= \frac{1}{\tilde{r}} \tilde{\partial}_\theta \langle X_0, -\sin \theta X_1 + \cos \theta X_2 \rangle \\
&= \frac{1}{\tilde{r}} \langle X_0, -\Upsilon \rangle + (-\sin \theta) \Upsilon_\theta \langle X_0, X_1 \rangle + \cos \theta \Upsilon_\theta \langle X_0, X_2 \rangle \\
&= \frac{2}{3} \varepsilon \phi R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta) + \frac{2}{3} R(\Upsilon_\theta, X_0, \eta, \Upsilon_\theta) - \frac{2}{3} R(\Upsilon, X_0, \eta, \Upsilon) \\
&\quad + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) + O(\varepsilon^2),
\end{aligned}$$

where we calculate  $\Upsilon_\theta \langle X_0, X_i \rangle$  directly from Lemma 2.2. To calculate  $\langle \nabla_{\Upsilon_\theta} X_0, \Upsilon_\theta \rangle$  we can regard  $\Upsilon_\theta$  as constant in  $(x_0, x_1, x_2)$  coordinate.

$$\begin{aligned}
\langle \nabla_{\Upsilon_\theta} X_0, \Upsilon_\theta \rangle &= \frac{1}{2} X_0 \langle \Upsilon_\theta, \Upsilon_\theta \rangle \\
&= O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta).
\end{aligned}$$

We have

$$\begin{aligned}
\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\phi + w)^2 \nabla_{\Upsilon_\theta} \Upsilon_\theta \rangle &= \frac{2}{3} \varepsilon^3 \dot{\phi} \phi^2 R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta) \\
&\quad + \frac{2}{3} \varepsilon^2 \dot{\phi} \phi (R(\Upsilon_\theta, X_0, \eta, \Upsilon_\theta) - R(\Upsilon, X_0, \eta, \Upsilon)) \\
&\quad + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4).
\end{aligned}$$

$$\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\frac{\partial w}{\partial \theta})^2 \nabla_{\Upsilon} \Upsilon \rangle = \varepsilon^3 Q(w, \eta).$$

$$\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\phi + w) \frac{\partial w}{\partial \theta} \nabla_{\Upsilon_\theta} \Upsilon \rangle = \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta).$$

$$\langle \frac{\dot{\phi}}{\phi} X_0, \varepsilon^2 (\phi + w) \frac{\partial w}{\partial \theta} \nabla_{\Upsilon} \Upsilon_\theta \rangle = \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta).$$

Note that  $\nabla_{\Upsilon_\theta} \Upsilon$  and  $\nabla_{\Upsilon} \Upsilon_\theta$  are not the same.

$$\langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon \frac{\partial w}{\partial \theta} \Upsilon_\theta \rangle = \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta).$$

$$\langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon \frac{\partial^2 w}{\partial \theta^2} \Upsilon \rangle = -\varepsilon \frac{\dot{\psi}}{\phi} \frac{\partial^2 w}{\partial \theta^2} + \varepsilon^3 L(w, \eta) + \varepsilon^3 Q(w, \eta).$$

$$\begin{aligned}
\langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon^2 (\phi + w)^2 \nabla_{\Upsilon_\theta} \Upsilon_\theta \rangle &= -\varepsilon^2 \frac{\dot{\psi}}{\phi} (\phi + w)^2 \langle \Upsilon, \nabla_{\Upsilon_\theta} \Upsilon_\theta \rangle \\
&= -\varepsilon^2 \frac{\dot{\psi}}{\phi} (\phi + w)^2 (\Upsilon_\theta \langle \Upsilon, \Upsilon_\theta \rangle - \langle \nabla_{\Upsilon_\theta} \Upsilon, \Upsilon_\theta \rangle),
\end{aligned}$$

First we have

$$\begin{aligned}
\Upsilon_\theta \langle \Upsilon, \Upsilon_\theta \rangle &= \frac{1}{\tilde{r}} (\langle \Upsilon_\theta, \Upsilon_\theta \rangle - \langle \Upsilon, \Upsilon \rangle) \\
&\quad + (-\sin \theta \cos \theta \Upsilon_\theta \langle X_1, X_1 \rangle + \sin \theta \cos \theta \Upsilon_\theta \langle X_2, X_2 \rangle \\
&\quad + (\cos^2 \theta - \sin^2 \theta) \Upsilon_\theta \langle X_1, X_2 \rangle) \\
&= \frac{1}{3} R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) + O(\varepsilon^2),
\end{aligned}$$

$$\begin{aligned}
\langle \nabla_{\Upsilon_\theta} \Upsilon, \Upsilon_\theta \rangle &= \frac{1}{\tilde{r}} \langle \nabla_{\tilde{r} \Upsilon_\theta} \Upsilon, \Upsilon_\theta \rangle \\
&= \frac{1}{\tilde{r}} \langle \nabla_{\Upsilon} \tilde{r} \Upsilon_\theta, \Upsilon_\theta \rangle \\
&= \frac{1}{\tilde{r}} \langle \Upsilon_\theta, \Upsilon_\theta \rangle + \langle \nabla_{\Upsilon} \Upsilon_\theta, \Upsilon_\theta \rangle.
\end{aligned}$$

From

$$\begin{aligned} \frac{1}{\tilde{r}} \langle \Upsilon_\theta, \Upsilon_\theta \rangle &= \frac{1}{\tilde{r}} + \frac{1}{3} \varepsilon \phi R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) + \frac{2}{3} R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) \\ &\quad + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) + O(\varepsilon^2), \end{aligned}$$

$$\begin{aligned} \langle \nabla_{\Upsilon} \Upsilon_\theta, \Upsilon_\theta \rangle &= \frac{1}{2} \Upsilon \langle \Upsilon_\theta, \Upsilon_\theta \rangle \\ &= \frac{1}{3} \varepsilon \phi R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) + \frac{1}{3} R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) \\ &\quad + O(\varepsilon^2) + \varepsilon L(w, \eta) + Q(w, \eta). \end{aligned}$$

We get

$$\begin{aligned} \langle \nabla_{\Upsilon_\theta} \Upsilon, \Upsilon_\theta \rangle &= \frac{1}{\tilde{r}} + \frac{2}{3} \varepsilon \phi R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) + R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) \\ &\quad + \varepsilon L(w, \eta) + \varepsilon^{-1} Q(w, \eta) + O(\varepsilon^2), \end{aligned}$$

$$\begin{aligned} \langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon^2 (\phi + w)^2 \nabla_{\Upsilon_\theta} \Upsilon_\theta \rangle &= \varepsilon (\phi^2 + \tau_0) + \varepsilon \frac{\dot{\psi}}{\phi} w + \frac{2}{3} \varepsilon^3 \phi^2 \dot{\psi} R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) \\ &\quad + \frac{2}{3} \phi \dot{\psi} R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) \\ &\quad + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4). \end{aligned}$$

$$\langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon^2 \left( \frac{\partial w}{\partial \theta} \right)^2 \nabla_{\Upsilon} \Upsilon \rangle = \varepsilon^3 Q(w, \eta).$$

$$\langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon^2 (\phi + w) \frac{\partial w}{\partial \theta} \nabla_{\Upsilon_\theta} \Upsilon \rangle = \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta).$$

$$\langle -\frac{\dot{\psi}}{\phi} \Upsilon, \varepsilon^2 (\phi + w) \frac{\partial w}{\partial \theta} \nabla_{\Upsilon} \Upsilon_\theta \rangle = \varepsilon^3 L(w, \eta) + \varepsilon^2 Q(w, \eta).$$

We collect all the terms and get

$$\begin{aligned} &\langle N_0, \nabla_{\partial_\theta} \partial_\theta \rangle \\ &= \varepsilon (\phi^2 + \tau_0) - \varepsilon \frac{\dot{\psi}}{\phi} \frac{\partial^2 w}{\partial \theta^2} + \varepsilon \frac{\dot{\psi}}{\phi} w \end{aligned} \tag{54}$$

$$\begin{aligned} &+ \frac{2}{3} \varepsilon^3 \phi^2 \dot{\psi} R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon^3 \dot{\phi} \phi^2 R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta) \\ &+ \frac{2}{3} \varepsilon^2 \phi \dot{\phi} R(\Upsilon_\theta, X_0, \eta, \Upsilon_\theta) - \frac{2}{3} \varepsilon^2 \phi \dot{\phi} R(\Upsilon, X_0, \eta, \Upsilon) \\ &+ \frac{2}{3} \phi \dot{\psi} R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4). \end{aligned} \tag{55}$$

For

$$\langle N, \nabla_{\partial_\theta} \partial_s \rangle = \langle N_0 + a_1 \partial_s + a_2 \partial_\theta, \nabla_{\partial_\theta} \partial_s \rangle$$

we don't need very precise expansion because  $g^{s\theta}$  is small relatively.

$$\langle \partial_s, \nabla_{\partial_\theta} \partial_s \rangle = O(\varepsilon^4) + \varepsilon^2 L(w, \eta) + \varepsilon^2 Q(w, \eta), \quad (56)$$

$$\langle \partial_\theta, \nabla_{\partial_\theta} \partial_s \rangle = \varepsilon^2 \phi \dot{\phi} + O(\varepsilon^4) + \varepsilon^2 L(w, \eta) + \varepsilon^2 Q(w, \eta). \quad (57)$$

$$\langle N_0, \nabla_{\partial_\theta} \partial_s \rangle = O(\varepsilon^3) + \varepsilon^2 L(w, \eta) + \varepsilon Q(w, \eta). \quad (58)$$

Now we can calculate the mean curvature

$$\begin{aligned} H(\mathcal{D}_{\phi_{\tau_0}, p_0, \varepsilon}(w, \eta)) &= g^{ss} \langle N, \nabla_{\partial_s} \partial_s \rangle + 2g^{s\theta} \langle N, \nabla_{\partial_\theta} \partial_s \rangle + g^{\theta\theta} \langle N, \nabla_{\partial_\theta} \partial_\theta \rangle \\ &= \frac{1}{k} (g^{ss} \langle kN, \nabla_{\partial_s} \partial_s \rangle + 2g^{s\theta} \langle kN, \nabla_{\partial_\theta} \partial_s \rangle \\ &\quad + g^{\theta\theta} \langle kN, \nabla_{\partial_\theta} \partial_\theta \rangle) \end{aligned}$$

From (9), (10), (11), (46), (47) and (51) we know

$$\begin{aligned} g^{ss} \langle kN, \nabla_{\partial_s} \partial_s \rangle &= \varepsilon^{-2} \phi^{-2} (1 - (\varepsilon^2 \dot{\psi}^2 R(\Upsilon, X_0, \Upsilon, X_0) + 2\varepsilon \phi^{-1} \dot{\psi}^2 R(\Upsilon, X_0, \eta, X_0) \\ &\quad + \frac{4}{3} \varepsilon \phi^{-1} \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \eta, \Upsilon) + 2\phi^{-2} \dot{\phi} \frac{\partial w}{\partial s} + 2\phi^{-2} \dot{\phi} \dot{\psi} \langle \Upsilon, \frac{\partial \eta}{\partial x_0} \rangle_e \\ &\quad + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3))) \\ &\quad (a_1 \frac{\varepsilon^2}{2} (2\phi \dot{\phi} + L(w, \eta) + \varepsilon L(\frac{\partial^2 \eta}{\partial x_0^2}) + Q(w, \eta) + O(\varepsilon^2)) \\ &\quad + a_2 \varepsilon^2 (O(\varepsilon^2) + L(w, \eta) + \varepsilon L(\frac{\partial^2 \eta}{\partial x_0^2}) + Q(w, \eta)) \\ &\quad + \varepsilon(\phi^2 - \tau_0) - \varepsilon \frac{\dot{\psi}}{\phi} \frac{\partial^2 w}{\partial s^2} - \varepsilon^2 \frac{\dot{\psi}^3}{\phi} \langle \frac{\partial^2 \eta}{\partial x_0^2}, \Upsilon \rangle - \varepsilon \frac{\dot{\psi} \ddot{\psi}}{\phi} \langle \frac{\partial \eta}{\partial x_0}, \Upsilon \rangle \\ &\quad + \varepsilon^3 (\phi \dot{\phi} \ddot{\psi} + 2\dot{\phi}^2 \dot{\psi} + \dot{\psi}^3) R(\Upsilon, X_0, \Upsilon, X_0) \\ &\quad + \varepsilon^2 (2\dot{\phi} \ddot{\psi} + 2\frac{\dot{\phi}^2 \dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi}) R(\Upsilon, X_0, \eta, X_0) \\ &\quad + \varepsilon^2 (\frac{2}{3} \dot{\phi} \ddot{\phi} + \frac{2}{3} \frac{\dot{\phi}^3}{\phi} - \frac{2}{3} \dot{\psi} \ddot{\psi}) R(\Upsilon, X_0, \eta, \Upsilon) \\ &\quad + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + \varepsilon^4 L(\partial^2 \eta) + \varepsilon^2 Q(w, \partial^2 \eta) + O(\varepsilon^4)) \end{aligned}$$

$$\begin{aligned}
&= \varepsilon^{-2}\phi^{-2}(a_1\varepsilon^2\phi\dot{\phi} + \varepsilon(\phi^2 - \tau_0) - \varepsilon\frac{\dot{\psi}}{\phi}\frac{\partial^2 w}{\partial s^2} - 2\varepsilon(\phi^2 - \tau_0)\frac{\dot{\phi}}{\phi^2}\frac{\partial w}{\partial s} \\
&\quad - \varepsilon^2\frac{\dot{\psi}^3}{\phi} < \frac{\partial^2 \eta}{\partial x_0^2}, \Upsilon > - \varepsilon(\frac{\dot{\psi}\ddot{\psi}}{\phi} + 2(\phi^2 - \tau_0)\frac{\dot{\phi}\dot{\psi}}{\phi^2}) < \frac{\partial \eta}{\partial x_0}, \Upsilon > \\
&\quad + \varepsilon^3(\phi\dot{\phi}\ddot{\psi} + 2\dot{\phi}^2\dot{\psi} + \dot{\psi}^3 - (\phi^2 - \tau_0)\dot{\psi}^2)R(\Upsilon, X_0, \Upsilon, X_0) \\
&\quad + \varepsilon^2(2\dot{\phi}\ddot{\psi} + 2\frac{\dot{\phi}^2\dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2\frac{\dot{\psi}^2}{\phi}(\phi^2 - \tau_0))R(\Upsilon, X_0, \eta, X_0) \\
&\quad + \varepsilon^2(\frac{2}{3}\dot{\phi}\ddot{\phi} + \frac{2}{3}\frac{\dot{\phi}^3}{\phi} - \frac{2}{3}\dot{\psi}\ddot{\psi} - \frac{4}{3}(\phi^2 - \tau_0)\frac{\dot{\phi}\dot{\psi}}{\phi})R(\Upsilon, X_0, \eta, \Upsilon) \\
&\quad + \varepsilon^3L(w, \eta) + \varepsilon Q(w, \eta) + \varepsilon^4L(\partial^2 \eta) + \varepsilon^2Q(w, \partial^2 \eta) + O(\varepsilon^4)
\end{aligned}$$

From (9), (10), (11), (52), (53) and (55) we know

$$\begin{aligned}
g^{\theta\theta} < kN, \nabla_{\partial_\theta} \partial_\theta > &= \varepsilon^{-2}\phi^{-2}(1 - (2\phi^{-1}w + \frac{1}{3}\varepsilon^2\phi^2R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) \\
&\quad + \frac{2}{3}\varepsilon\phi R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) + \varepsilon^2L(w, \eta) + Q(w, \eta) + O(\varepsilon^3))) \\
&\quad (a_1(-\varepsilon^2\phi\dot{\phi} + O(\varepsilon^4) + \varepsilon^2L(w, \eta) + \varepsilon^2Q(w, \eta)) \\
&\quad + a_2(O(\varepsilon^4) + \varepsilon^2L(w, \eta) + \varepsilon^2Q(w, \eta)) \\
&\quad + \varepsilon(\phi^2 + \tau_0) - \varepsilon\frac{\dot{\psi}}{\phi}\frac{\partial^2 w}{\partial \theta^2} + \varepsilon\frac{\dot{\psi}}{\phi}w + \frac{2}{3}\varepsilon^3\phi^2\dot{\psi}R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) \\
&\quad + \frac{2}{3}\varepsilon^3\dot{\phi}\phi^2R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta) + \frac{2}{3}\varepsilon^2\phi\dot{\phi}R(\Upsilon_\theta, X_0, \eta, \Upsilon_\theta) \\
&\quad - \frac{2}{3}\varepsilon^2\phi\dot{\phi}R(\Upsilon, X_0, \eta, \Upsilon) + \frac{2}{3}\varepsilon^2\phi\dot{\psi}R(\Upsilon, \Upsilon_\theta, \eta, \Upsilon_\theta) \\
&\quad + \varepsilon^3L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4)) \\
&= \varepsilon^{-2}\phi^{-2}(-\varepsilon^2a_1\phi\dot{\phi} + \varepsilon(\phi^2 + \tau_0) - \varepsilon\frac{\dot{\psi}}{\phi}\frac{\partial^2 w}{\partial \theta^2} - \varepsilon\frac{\dot{\psi}}{\phi}w \\
&\quad + \frac{1}{3}\varepsilon^3\phi^2\dot{\psi}R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) + \frac{2}{3}\varepsilon^3\dot{\phi}\phi^2R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta) \\
&\quad + \frac{2}{3}\varepsilon^2\phi\dot{\phi}R(\Upsilon_\theta, X_0, \eta, \Upsilon_\theta) - \frac{2}{3}\varepsilon^2\phi\dot{\phi}R(\Upsilon, X_0, \eta, \Upsilon) \\
&\quad + \varepsilon^3L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4))
\end{aligned}$$

From (9), (10), (11), (56), (57) and (58) we know

$$\begin{aligned}
g^{s\theta} < kN, \nabla_{\partial_\theta} \partial_s > &= \varepsilon^{-2} \phi^{-2} \left( -\left( \frac{2}{3} \varepsilon^2 \dot{\phi} \dot{\psi} R(\Upsilon, X_0, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon \dot{\psi} (R(\Upsilon, X_0, \eta, \Upsilon_\theta) \right. \right. \\
&\quad \left. \left. + R(\eta, X_0, \Upsilon, \Upsilon_\theta)) + \frac{1}{3} \varepsilon \dot{\phi} R(\eta, \Upsilon, \Upsilon, \Upsilon_\theta) \right. \right. \\
&\quad \left. \left. + \phi^{-2} \dot{\phi} \frac{\partial w}{\partial \theta} + \phi^{-1} \dot{\psi} < \frac{\partial \eta}{\partial x_0}, \Upsilon_\theta >_e + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3) \right) \right) \\
&\quad (a_1(O(\varepsilon^4) + \varepsilon^2 L(w, \eta) + \varepsilon^2 Q(w, \eta)) \\
&\quad + a_2(\varepsilon^2 \dot{\phi} \dot{\phi} + O(\varepsilon^4) + \varepsilon^2 L(w, \eta) + \varepsilon^2 Q(w, \eta)) \\
&\quad + O(\varepsilon^3) + \varepsilon^2 L(w, \eta) + \varepsilon Q(w, \eta)) \\
&= \varepsilon^{-2} \phi^{-2} (\varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + O(\varepsilon^4)).
\end{aligned}$$

So

$$\begin{aligned}
H &= \frac{1}{k} \varepsilon^{-2} \phi^{-2} (2\varepsilon \phi^2 - \varepsilon \frac{\dot{\psi}}{\phi} (\frac{\partial^2 w}{\partial s^2} + \frac{\partial^2 w}{\partial \theta^2}) - 2\varepsilon (\phi^2 - \tau_0) \frac{\dot{\phi}}{\phi^2} \frac{\partial w}{\partial s} - \varepsilon \frac{\dot{\psi}}{\phi} w \\
&\quad - \varepsilon^2 \frac{\dot{\psi}^3}{\phi} < \frac{\partial^2 \eta}{(\partial x^0)^2}, \Upsilon > - \varepsilon (\frac{\dot{\psi} \ddot{\psi}}{\phi} + 2(\phi^2 - \tau_0) \frac{\dot{\phi} \dot{\psi}}{\phi^2}) < \frac{\partial \eta}{\partial x^0}, \Upsilon > \\
&\quad + \varepsilon^3 (\phi \dot{\phi} \ddot{\psi} + 2\dot{\phi}^2 \dot{\psi} + \dot{\psi}^3 - (\phi^2 - \tau_0) \dot{\psi}^2) R(\Upsilon, X_0, \Upsilon, X_0) \\
&\quad + \frac{1}{3} \varepsilon^3 \phi^2 \dot{\psi} R(\Upsilon, \Upsilon_\theta, \Upsilon, \Upsilon_\theta) + \frac{2}{3} \varepsilon^3 \dot{\phi} \phi^2 R(\Upsilon_\theta, X_0, \Upsilon, \Upsilon_\theta) \\
&\quad + \varepsilon^2 (2\phi \ddot{\psi} + 2 \frac{\dot{\phi}^2 \dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2 \frac{\dot{\psi}^2}{\phi} (\phi^2 - \tau_0)) R(\Upsilon, X_0, \eta, X_0) \\
&\quad + \varepsilon^2 (\frac{2}{3} \phi \ddot{\phi} + \frac{2}{3} \frac{\dot{\phi}^3}{\phi} - \frac{2}{3} \dot{\psi} \ddot{\psi} - \frac{4}{3} (\phi^2 - \tau_0) \frac{\dot{\phi} \dot{\psi}}{\phi} - \frac{2}{3} \phi \dot{\phi}) R(\Upsilon, X_0, \eta, \Upsilon) \\
&\quad + \frac{2}{3} \varepsilon^2 \phi \dot{\phi} R(\Upsilon_\theta, X_0, \eta, \Upsilon_\theta) \\
&\quad + \varepsilon^3 L(w, \eta) + \varepsilon Q(w, \eta) + \varepsilon^4 L(\partial^2 \eta) + \varepsilon^2 Q(w, \partial^2 \eta) + O(\varepsilon^4)
\end{aligned}$$

From (12) we know

$$\begin{aligned}
\frac{1}{k} &= 1 - \frac{\varepsilon^2}{2} \dot{\phi}^2 R(\Upsilon, X_0, \Upsilon, X_0) - \varepsilon \frac{\dot{\phi}^2}{\phi} R(\Upsilon, X_0, \eta, X_0) \\
&\quad + \frac{2}{3} \varepsilon \frac{\dot{\phi} \dot{\psi}}{\phi} R(\Upsilon, X_0, \eta, \Upsilon) + \varepsilon^2 L(w, \eta) + Q(w, \eta) + O(\varepsilon^3).
\end{aligned}$$

So at last we get (13).



## B The proof of Lemma 3.1 and Lemma 3.5

The proof of Lemma 3.1. The goal is to prove that

$$\begin{aligned} & \int_{a_1}^{b_1} \frac{1}{\phi} (2\dot{\phi}\ddot{\psi} + 2\frac{\dot{\phi}^2\dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2\frac{\dot{\psi}^2}{\phi}(\phi^2 - \tau_0) - 2\phi\dot{\phi}^2) d\psi \cdot \int_{a_1}^{b_1} \frac{\phi^2}{\dot{\psi}^3} d\psi \\ &= (b_1 - a_1)^2, \end{aligned}$$

where  $[a_1, b_1]$  is one period for  $\phi(\psi)$ .

*Proof.*

$$\int_{a_1}^{b_1} \frac{\phi^2}{\dot{\psi}^3} d\psi = \int_{a_1}^{b_1} \frac{1}{\phi} (1 + \phi_\psi^2)^{\frac{3}{2}} d\psi.$$

From

$$\phi_{\psi\psi} - \phi^{-1}(1 + \phi_\psi^2) + 2(1 + \phi_\psi^2)^{\frac{3}{2}} = 0$$

we have

$$\int_{a_1}^{b_1} \frac{\phi^2}{\dot{\psi}^3} d\psi = \int_{a_1}^{b_1} \left( \frac{1 + \phi_\psi^2}{2\phi^2} - \frac{\phi_{\psi\psi}}{2\phi} \right) d\psi.$$

Note that

$$\begin{aligned} \int_{a_1}^{b_1} -\frac{\phi_{\psi\psi}}{2\phi} d\psi &= -\frac{1}{2} \left( \frac{\phi_\psi}{2\phi} \Big|_{a_1}^{b_1} - \int_{a_1}^{b_1} \phi_\psi \frac{-\phi_\psi}{\phi^2} d\psi \right) \\ &= -\frac{1}{2} \int_{a_1}^{b_1} \phi_\psi \frac{-\phi_\psi}{\phi^2} d\psi. \end{aligned}$$

So we have

$$\int_{a_1}^{b_1} \frac{\phi^2}{\dot{\psi}^3} d\psi = \int_{a_1}^{b_1} \frac{1}{2\phi^2} d\psi.$$

Also from direct computation one can get

$$\begin{aligned} & \int_{a_1}^{b_1} \frac{1}{\phi} (2\dot{\phi}\ddot{\psi} + 2\frac{\dot{\phi}^2\dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2\frac{\dot{\psi}^2}{\phi}(\phi^2 - \tau_0) - 2\phi\dot{\phi}^2) d\psi \\ &= \int_{a_1}^{b_1} (\phi^2 (2\phi_{\psi\psi} (1 + \phi_\psi^2)^{-\frac{5}{2}} - (1 + \phi_\psi^2)^{-\frac{5}{2}} \phi_{\psi\psi} \phi_\psi^2) \\ &\quad + \phi (3(1 + \phi_\psi^2)^{-\frac{3}{2}} \phi_\psi^2 + (1 + \phi_\psi^2)^{-\frac{3}{2}})) d\psi \\ &= \int_{a_1}^{b_1} (2\phi^2 + \frac{2\phi^2 \phi_{\psi\psi}}{(1 + \phi_\psi^2)^{\frac{3}{2}}}) d\psi. \end{aligned}$$

We assume that

$$\phi(s) = \sqrt{\tau_0} \exp(\sigma(s))$$

then we have

$$\begin{aligned}\dot{\psi} &= \sqrt{\tau_0} \exp(\sigma(s)) \sqrt{1 - \sigma_s^2} \\ \phi_\psi &= \frac{\sigma_s}{\sqrt{1 - \sigma_s^2}} \\ \phi_{\psi\psi} &= \frac{\sigma_{ss}}{\sqrt{\tau_0} \exp(\sigma)(1 - \sigma_s^2)^2}\end{aligned}$$

Also we have

$$\begin{aligned}1 - \sigma_s^2 &= 4\tau_0 \cosh^2 \sigma \\ \sigma_{ss} &= -2\tau_0 \sinh 2\sigma.\end{aligned}$$

For  $\phi$  there are two particular points in one period such that  $\sigma = 0$ . Suppose these two points are

$$s = s_1, s = s_2$$

and suppose  $[s_1, s_3]$  is one period for  $\phi(s)$ . Note that  $s_2 \in (s_1, s_3)$ .

Here we use one particular property of Delaunay surface

$$\sigma(s_2 - t) = -\sigma(s_2 + t).$$

Then from direct calculation we know

$$\begin{aligned}\int_{a_1}^{b_1} \frac{1}{2\phi^2} d\phi &= \int_{s_1}^{s_3} \cosh^2 \sigma ds \\ \int_{a_1}^{b_1} (2\phi^2 + \frac{2\phi^2 \phi_{\psi\psi}}{(1 + \phi_\psi^2)^{\frac{3}{2}}}) d\psi &= 4\tau_0^2 \int_{s_1}^{s_3} \cosh^2 \sigma ds\end{aligned}$$

and

$$b_1 - a_1 = 2\tau_0 \int_{s_1}^{s_3} \cosh^2 \sigma ds.$$

So we know

$$\begin{aligned}& \int_{a_1}^{b_1} \frac{1}{\phi} (2\phi\ddot{\psi} + 2\frac{\dot{\phi}^2 \dot{\psi}}{\phi} + \frac{\dot{\psi}^3}{\phi} - 2\frac{\dot{\psi}^2}{\phi} (\phi^2 - \tau_0) - 2\phi\dot{\phi}^2) d\psi \cdot \int_{a_1}^{b_1} \frac{\phi^2}{\dot{\psi}^3} d\psi \\ &= (b_1 - a_1)^2.\end{aligned}$$

□

Proof of Lemma 3.5. Without loss of generality, we may assume  $\tau = \tau_0$  and  $a_\tau = a_1, b_\tau = b_1$ . The goal is to prove

$$\int_{a_1}^{b_1} \phi \left( \frac{\partial \phi}{\partial \psi} \right) \phi^{-2} \left( \frac{2}{3} \dot{\phi} \ddot{\phi} + \frac{2}{3} \frac{\dot{\phi}^3}{\phi} - \frac{2}{3} \dot{\psi} \ddot{\psi} - \frac{4}{3} (\phi^2 - \tau_0) \frac{\dot{\phi} \dot{\psi}}{\phi} + \frac{4}{3} \phi \dot{\phi} \dot{\psi} \right) d\psi = 0$$

which is equivalent to

$$\int_{a_1}^{b_1} \phi_\psi^2 \phi^{-2} \dot{\psi} (\phi \ddot{\phi} + \dot{\phi}^2 - 2(\phi^2 + \tau_0)(\phi^2 - \tau_0)) d\psi = 0.$$

*Proof.* Note that

$$\begin{aligned}
& \int_{a_1}^{b_1} \phi_\psi^2 \phi^{-2} \dot{\psi} (\phi \ddot{\phi} + \dot{\phi}^2 - 2(\phi^2 + \tau_0)(\phi^2 - \tau_0)) d\psi = 0 \\
&= \int_{a_1}^{b_1} \frac{\sigma_s^2}{1 - \sigma_s^2} \frac{1}{\tau \exp(2\sigma)} \tau_0 \exp(2\sigma) (1 - \sigma_s^2) (\phi \ddot{\phi} + \dot{\phi}^2 - 2(\phi^2 + \tau_0)(\phi^2 - \tau_0)) d\psi \\
&= \int_{a_1}^{b_1} \sigma_s^2 (\phi \ddot{\phi} + \dot{\phi}^2 - 2(\phi^2 + \tau_0)(\phi^2 - \tau_0)) d\psi.
\end{aligned}$$

$$\begin{aligned}
& \int_{a_1}^{b_1} \sigma_s^2 (\phi \ddot{\phi} + \dot{\phi}^2) d\psi \\
&= \int_{a_1}^{b_1} \sigma_s^2 (\phi \dot{\phi})_s ds \\
&= - \int_{a_1}^{b_1} \phi \dot{\phi} 2\sigma_s \sigma_{ss} ds \\
&= -2 \int_{a_1}^{b_1} \exp(\sigma) \exp(\sigma) \sigma_s^2 \sigma_{ss} ds \\
&= 4\tau_0^2 \int_{a_1}^{b_1} \exp(2\sigma) (1 - 4\tau_0 \cosh^2(\sigma)) \sinh(2\sigma) ds \\
&= 2\tau_0^2 \int_{a_1}^{b_1} (1 - 4\tau_0 \cosh^2(\sigma)) \sinh^2(2\sigma) ds.
\end{aligned}$$

$$\begin{aligned}
& -2 \int_{a_1}^{b_1} \sigma_s^2 (\phi^2 + \tau_0)(\phi^2 - \tau_0) ds \\
&= -2 \int_{a_1}^{b_1} \sigma_s^2 (\tau_0 \exp(2\sigma) + \tau_0)(\tau_0 \exp(2\sigma) - \tau_0) ds \\
&= -2\tau_0^2 \int_{a_1}^{b_1} \sigma_s^2 4 \sinh(\sigma) \cosh(\sigma) \exp(2\sigma) ds \\
&= -2\tau_0^2 \int_{a_1}^{b_1} \sigma_s^2 2 \sinh(2\sigma) \exp(2\sigma) ds \\
&= -2\tau_0^2 \int_{a_1}^{b_1} \sigma_s^2 \sinh^2(2\sigma) ds \\
&= -2\tau_0^2 \int_{a_1}^{b_1} (1 - 4\tau_0 \cosh^2(\sigma)) \sinh^2(2\sigma) ds.
\end{aligned}$$

So we proved that

$$\int_{a_1}^{b_1} \phi_\psi^2 \phi^{-2} \dot{\psi} (\phi \ddot{\phi} + \dot{\phi}^2 - 2(\phi^2 + \tau_0)(\phi^2 - \tau_0)) d\psi = 0.$$

□

## C The proof of Lemma 3.10

First we prove

$$|\Phi'(\psi) - 1| \leq C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|).$$

Note that

$$\Phi'(\psi) = \frac{d\Phi(\psi)}{ds_0} \frac{ds_0}{dl} \frac{dl}{d\psi}$$

where

$$dl = \sqrt{\langle \partial_{s_0}, \partial_{s_0} \rangle ds_0^2 + \langle \partial_\tau, \partial_\tau \rangle d\tau^2}$$

is the arc length of  $(\phi(\psi), \zeta(\psi))$ . On the curve  $(\phi(\psi), \zeta(\psi))$  we have

$$1 = \sqrt{\langle \partial_{s_0}, \partial_{s_0} \rangle \left(\frac{ds_0}{dl}\right)^2 + \langle \partial_\tau, \partial_\tau \rangle \left(\frac{d\tau}{d\psi}\right)^2 \left(\frac{d\psi}{dl}\right)^2}. \quad (59)$$

In  $C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|)$  neighborhood of  $(\phi_{\tau(0)}(\psi), \zeta_{\tau(0)}(\psi))$ ,

$$|\langle \partial_{s_0}, \partial_{s_0} \rangle - 1| \leq C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|). \quad (60)$$

And

$$\left|\frac{d\tau}{d\psi}\right| \leq C\varepsilon(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|). \quad (61)$$

Note that

$$\begin{aligned} \frac{dl}{d\psi} &= \sqrt{\left(\frac{d\phi}{d\psi}\right)^2 + \left(\frac{d\zeta}{d\psi}\right)^2} \\ &= \sqrt{\zeta^2 + (\phi^{-1}(1 + \phi^2) - (2 + \rho)(1 + \zeta^2)^{\frac{3}{2}})^2} \end{aligned}$$

and

$$\begin{aligned} \frac{ds_0}{d\Phi(\psi)} &= \sqrt{\left(\frac{d\phi_{\tau(0)}}{d\Phi(\psi)}\right)^2 + \left(\frac{d\zeta_{\tau(0)}}{d\Phi(\psi)}\right)^2} \\ &= \sqrt{\zeta_{\tau(0)}^2 + (\phi_{\tau(0)}^{-1}(1 + \phi_{\tau(0)}^2) - 2(1 + \zeta_{\tau(0)}^2)^{\frac{3}{2}})^2} \Big|_{\Phi(\psi)}. \end{aligned}$$

So we have

$$\left|\frac{dl}{d\psi} / \frac{ds_0}{d\Phi(\psi)} - 1\right| \leq C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|) \quad (62)$$

Note that  $\frac{ds_0}{d\Phi(\psi)}$  has both positive upper bound and positive lower bound which only depend on  $\tau_0$  and so does  $\frac{dl}{d\psi}$  when  $\varepsilon$  is sufficiently small.

So we know  $\left(\frac{d\psi}{dl}\right)^2$  is bounded and together with (59)(60)(61) we get

$$\left|\frac{ds_0}{dl} - 1\right| \leq C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|). \quad (63)$$

From (62)(63), we get

$$\begin{aligned} |\Phi'(\psi) - 1| &= \left| \frac{\sqrt{\zeta^2 + (\phi^{-1}(1 + \zeta^2) - (2 + \rho)(1 + \zeta^2)^{\frac{3}{2}})^2} |_{\psi}}{\sqrt{\zeta_{\tau(0)}^2 + (\phi_{\tau(0)}^{-1}(1 + \phi_{\tau(0)}^2) - 2(1 + \zeta_{\tau(0)}^2)^{\frac{3}{2}})^2} |_{\Phi(\psi)}} \frac{ds_0}{dl} - 1 \right| \\ &\leq C(\tau_0)(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|). \end{aligned}$$

By integration we know that

$$|\Phi(\psi) - \psi| \leq \frac{C(\tau_0)}{\varepsilon}(\varepsilon + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|).$$

## D The proof of Lemma 3.14

$$\begin{aligned} &\begin{pmatrix} \beta_1(\psi_i) & \beta_2(\psi_i) \\ \frac{\partial \beta_1}{\partial \psi}(\psi_i) & \frac{\partial \beta_2}{\partial \psi}(\psi_i) \end{pmatrix} \\ &= \begin{pmatrix} 1 + e_{11}^i & e_{12}^i \\ \kappa + e_{21}^i & 1 + e_{22}^i \end{pmatrix} \cdots \begin{pmatrix} 1 + e_{11}^1 & e_{12}^1 \\ \kappa + e_{21}^1 & 1 + e_{22}^1 \end{pmatrix} \\ &= \begin{pmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{pmatrix} \end{aligned}$$

where  $|e_{kl}^i| \leq C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|)$ .

**Lemma D.1.** *If  $|a_{ij}| \leq C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|)$  and for  $k \in \mathbb{Z}, k \leq \frac{C(\tau_0)}{\varepsilon}$ ,  $|f| \leq kC(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|)$ , there are  $\tilde{a}_{11}, \tilde{a}_{21}, \tilde{a}_{22}, \tilde{f}$  such that the following holds*

$$\begin{pmatrix} 1 + a_{11} & a_{12} \\ \kappa + a_{21} & 1 + a_{22} \end{pmatrix} \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \tilde{f} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + \tilde{a}_{11} & 0 \\ \kappa + \tilde{a}_{21} & 1 + \tilde{a}_{22} \end{pmatrix}, \quad (64)$$

with

$$\begin{aligned} |\tilde{f}| &\leq (k + 2)C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|), \\ |\tilde{a}_{11}|, |\tilde{a}_{22}| &\leq (\kappa k + 1)C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|). \end{aligned}$$

*Proof.* (of Lemma D.1) Note that

$$\begin{aligned} |(\kappa + a_{21})f| &\leq \frac{C(\tau_0)}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2 \|\mu\|_{C^0} + \varepsilon^2 |\omega|) \\ &\leq C(\tau_0)(1 + C_1 + C_2 + C_3)\varepsilon. \end{aligned}$$

So we can choose  $\varepsilon$  sufficiently small such that

$$1 + a_{22} + f(\kappa + a_{21}) \geq 1 - C(\tau_0, C_1, C_2, C_3)\varepsilon$$

and it is invertible. Then one can check directly that

$$\begin{cases} \tilde{a}_{11} &= a_{11} - (\kappa + a_{21}) \frac{(1+a_{11})f+a_{12}}{1+a_{22}+f(\psi_1+a_{21})}, \\ \tilde{a}_{21} &= a_{21}, \\ \tilde{a}_{22} &= (\kappa + a_{21})f + a_{22}, \\ \tilde{f} &= \frac{(1+a_{11})f+a_{12}}{1+a_{22}+f(\kappa+a_{21})} \end{cases}$$

satisfies (64). So one can easily get

$$\begin{aligned} |\tilde{f}| &\leq (k+2)C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|), \\ |\tilde{a}_{11}|, |\tilde{a}_{22}| &\leq (\kappa k + 1)C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|). \end{aligned}$$

□

Note that

$$\begin{aligned} &\begin{pmatrix} 1 + e_{11}^i & e_{12}^i \\ \kappa + e_{21}^i & 1 + e_{22}^i \end{pmatrix} \cdots \begin{pmatrix} 1 + e_{11}^1 & e_{12}^1 \\ \kappa + e_{21}^1 & 1 + e_{22}^1 \end{pmatrix} \\ &= \begin{pmatrix} 1 + e_{11}^i & e_{12}^i \\ \kappa + e_{21}^i & 1 + e_{22}^i \end{pmatrix} \cdots \begin{pmatrix} 1 + e_{11}^2 & e_{12}^2 \\ \kappa + e_{21}^2 & 1 + e_{22}^2 \end{pmatrix} \\ &\begin{pmatrix} 1 & -f_{12}^1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + \tilde{e}_{11}^1 & 0 \\ \kappa + \tilde{e}_{21}^1 & 1 + \tilde{e}_{22}^1 \end{pmatrix}. \end{aligned}$$

Because  $i$  is at most as big as  $\frac{L_\Gamma}{\varepsilon\psi_1}$  and  $|f_{12}^1|, |e_{ij}^k| \leq C(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|)$ , we can use Lemma D.1 for  $i$  times. Note that by induction

$$|f_{12}^j| \leq 2jC(\tau_0)(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|)$$

and  $2j \leq \frac{2L_\Gamma}{\varepsilon\psi_1} \leq \frac{C(\tau_0)}{\varepsilon}$ . At last we get

$$\begin{aligned} &\begin{pmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{pmatrix} \\ &= \begin{pmatrix} 1 & -f_{12}^i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 + \tilde{e}_{11}^i & 0 \\ \kappa + \tilde{e}_{21}^i & 1 + \tilde{e}_{22}^i \end{pmatrix} \cdots \begin{pmatrix} 1 + \tilde{e}_{11}^1 & 0 \\ \kappa + \tilde{e}_{21}^1 & 1 + \tilde{e}_{22}^1 \end{pmatrix} \end{aligned}$$

where

$$|f_{12}^i|, |\tilde{e}_{kl}^j| \leq \frac{C(\tau_0)}{\varepsilon}(\varepsilon^2 + \|\xi\|_{C^0} + \varepsilon^2\|\mu\|_{C^0} + \varepsilon^2|\omega|).$$

If we replace  $-f_{12}^i$  and  $\tilde{e}_{kl}^j$  by  $Er = \varepsilon C(\tau_0)(1 + C_1 + C_2 + C_3)$  then all  $A_{kl}^i$  will become bigger. Note that

$$\begin{aligned} &\begin{pmatrix} 1 + Er & 0 \\ \kappa + Er & 1 + Er \end{pmatrix} \cdots \begin{pmatrix} 1 + Er & 0 \\ \kappa + Er & 1 + Er \end{pmatrix} \\ &= \begin{pmatrix} (1 + Er)^i & 0 \\ i(\kappa + Er)(1 + Er)^{i-1} & (1 + Er)^i \end{pmatrix} \end{aligned}$$

Note that  $i \leq \frac{C(\tau_0)}{\varepsilon}$ . So

$$(1 + Er)^i \leq \exp C(\tau_0)(1 + C_1 + C_2 + C_3)$$

$$i(\kappa + Er)(1 + Er)^{i-1} \leq \frac{1}{\varepsilon} \exp C(\tau_0)(1 + C_1 + C_2 + C_3)$$

In the same way if we replace  $-f_{12}^i$  all  $\tilde{e}_{kl}^j$  by  $-Er$  then all  $A_{kl}^i$  will become smaller. Note that

$$\begin{pmatrix} 1 - Er & 0 \\ \kappa - Er & 1 - Er \end{pmatrix} \cdots \begin{pmatrix} 1 - Er & 0 \\ \kappa - Er & 1 - Er \end{pmatrix}$$

$$= \begin{pmatrix} (1 - Er)^i & 0 \\ i(\kappa - Er)(1 - Er)^{i-1} & (1 - Er)^i \end{pmatrix}$$

and

$$(1 - Er)^i \geq \exp(-C(\tau_0)(1 + C_1 + C_2 + C_3))$$

$$i(\kappa - Er)(1 - Er)^{i-1} \geq \frac{1}{\varepsilon} \exp(-C(\tau_0)(1 + C_1 + C_2 + C_3))$$

So we know

$$\begin{aligned} \exp(-C(\tau_0)(1 + C_1 + C_2 + C_3)) &\leq A_{22}^i \leq \exp C(\tau_0)(1 + C_1 + C_2 + C_3), \\ \exp(-C(\tau_0)(1 + C_1 + C_2 + C_3))\varepsilon^{-1} &\leq A_{21}^i \leq \exp C(\tau_0)(1 + C_1 + C_2 + C_3)\varepsilon^{-1}, \\ |A_{12}^i| &\leq C(\tau_0, C_1, C_2, C_3)\varepsilon \quad \exp C(\tau_0)(1 + C_1 + C_2 + C_3). \end{aligned}$$

There is another way we can do this (through similar argument)

$$\begin{pmatrix} A_{11}^i & A_{12}^i \\ A_{21}^i & A_{22}^i \end{pmatrix}$$

$$= \begin{pmatrix} 1 + e_{11}^i & e_{12}^i \\ \kappa + e_{21}^i & 1 + e_{22}^i \end{pmatrix} \cdots \begin{pmatrix} 1 + e_{11}^1 & e_{12}^1 \\ \kappa + e_{21}^1 & 1 + e_{22}^1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 + \bar{e}_{11}^i & \bar{e}_{12}^i \\ \kappa + \bar{e}_{21}^i & 1 + \bar{e}_{22}^i \end{pmatrix} \begin{pmatrix} 1 & h_{12}^1 \\ 0 & 1 \end{pmatrix} \cdots \begin{pmatrix} 1 + e_{11}^1 & e_{12}^1 \\ \kappa + e_{21}^1 & 1 + e_{22}^1 \end{pmatrix}$$

$$= \cdots$$

$$= \begin{pmatrix} 1 + \bar{e}_{11}^i & \bar{e}_{12}^i \\ \kappa + \bar{e}_{21}^i & 1 + \bar{e}_{22}^i \end{pmatrix} \cdots \begin{pmatrix} 1 + \bar{e}_{11}^1 & \bar{e}_{12}^1 \\ \kappa + \bar{e}_{21}^1 & 1 + \bar{e}_{22}^1 \end{pmatrix} \begin{pmatrix} 1 & h_{12}^i \\ 0 & 1 \end{pmatrix}.$$

And we can prove that

$$\exp(-(C + C_1 + C_2 + C_3)) \leq A_{11}^i \leq \exp(C + C_1 + C_2 + C_3).$$

## E The proof of Lemma 3.16

1.  $\beta_\mu$  estimate From

$$\frac{d}{dt} \phi_{\xi, \mu + t\Delta\mu, \omega, \tau(0)}(\psi)|_{t=0} = \beta_\mu(\psi)$$

we have

$$\begin{cases} \mathcal{L}_{\xi, \mu, \omega, \tau(0)} \beta_\mu(\psi) = -\varepsilon^3 (1 + \phi_\psi^2)^{\frac{3}{2}} \Delta \mu, \\ \beta_\mu(0) = 0, \\ \beta'_\mu(0) = 0. \end{cases} \quad (65)$$

We have

$$\begin{aligned} \beta_\mu(\psi) &= -\varepsilon^3 \int_0^\psi R(t)^{-1} (\beta_2(\psi) \beta_1(t) - \beta_1(\psi) \beta_2(t)) (1 + \phi_\psi^2(t))^{\frac{3}{2}} \Delta \mu(t) dt, \\ \beta'_\mu(\psi) &= -\varepsilon^3 \int_0^\psi R(t)^{-1} (\beta'_2(\psi) \beta_1(t) - \beta'_1(\psi) \beta_2(t)) (1 + \phi_\psi^2(t))^{\frac{3}{2}} \Delta \mu(t) dt. \end{aligned}$$

So from (34)

$$\|\beta_\mu(\psi)\|_{C_\varepsilon^1} \leq C(\tau_0, C_1, C_2, C_3) \varepsilon \|\Delta \mu\|_{C^0}.$$

**2.  $\beta_\xi$  estimate** From

$$\frac{d}{dt} \phi_{\xi+t\Delta\xi, \mu, \omega, \tau(0)}(\psi)|_{t=0} = \beta_\xi(\psi)$$

$$\begin{cases} \mathcal{L}_{\xi, \mu, \omega, \tau(0)} \beta_\xi(\psi) = -\varepsilon F_4(\phi, \phi_\psi) (1 + \phi_\psi^2)^{\frac{3}{2}} \Delta \xi \\ \beta_\xi(0) = 0, \\ \beta'_\xi(0) = 0. \end{cases} \quad (66)$$

It is easy to prove that

$$\|\beta_\xi(\psi)\|_{C_\varepsilon^1} \leq \frac{C(\tau_0, C_1, C_2, C_3)}{\varepsilon} \|\Delta \xi\|_{C^0}.$$

We are going to prove (38).

Consider

$$\begin{aligned} & |(\beta_\xi, \beta'_\xi) \cdot (\tau_\phi, \tau_\zeta)|_{\bar{\psi}}| \\ &= \left| \frac{d}{dt} \tau|_{\bar{\psi}} \right| \\ &= \left| \frac{d}{dt} \int_0^{\bar{\psi}} \phi \phi_\psi \rho d\psi \right| \\ &= \left| \int_0^{\bar{\psi}} \varepsilon^2 \left( \frac{\partial \hat{F}_1}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_1}{\partial \zeta} \beta'_\xi \right) + \varepsilon^3 (\phi_\psi \beta_\xi + \phi \beta'_\xi) \mu \right. \\ &\quad \left. + \varepsilon \left( \frac{\partial \hat{F}_2}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_2}{\partial \zeta} \beta'_\xi \right) \xi + \varepsilon \hat{F}_2 \Delta \xi + \varepsilon^3 \omega \left( \frac{\partial \hat{F}_3}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_3}{\partial \zeta} \beta'_\xi \right) d\psi \right| \\ &\leq C(\tau_0, C_1, C_2, C_3) \|\Delta \xi\|_{C^0}, \end{aligned}$$



where

$$\begin{aligned}\hat{F}_1 &= \phi \frac{\partial \phi}{\partial \psi} F_1(\phi, \phi_\psi) \star R_1, \\ \hat{F}_2 &= \phi \frac{\partial \phi}{\partial \psi} F_4(\phi, \phi_\psi), \\ \hat{F}_3 &= \phi_\psi^2.\end{aligned}$$

On the points  $\psi_i$  where  $\phi$  attains its local minimum, we have  $\tau_\zeta = 0$  and  $|\tau_\phi|$  has uniform positive lower bound which only depends on  $\tau_0$ . So

$$\begin{aligned}|\beta_\xi(\psi_i)| &\leq C(\tau_0, C_1, C_2, C_3) \|\Delta\xi\|_{C^0} \\ |\beta'_\xi(\psi_i)| &\leq \frac{C(\tau_0, C_1, C_2, C_3)}{\varepsilon} \|\Delta\xi\|_{C^0}.\end{aligned}$$

So from (35), when  $\psi \in [\psi_i, \psi_{i+1}]$ ,

$$\begin{aligned}& \|\beta_\xi(\psi) - [\beta_\xi(\psi_i)h(\tau(0))((\psi - \psi_i)\frac{\partial \phi}{\partial \psi} + v_i(\psi)) \\ & + \beta'_\xi(\psi_i)h(\tau(0))\frac{\partial \phi}{\partial \psi}]\|_{C^1_\varepsilon} \\ & \leq C(\tau_0, C_1, C_2, C_3)\varepsilon \|\Delta\xi\|_{C^0}.\end{aligned}\tag{67}$$

This tells us that the dominant part of  $\beta_\xi(\psi)$  is  $\beta'_\xi(\psi_i)h(\tau(0))\frac{\partial \phi}{\partial \psi}$ .

Then we have, for  $\bar{\psi} \in [\psi_k, \psi_{k+1})$

$$\begin{aligned}& \left| \int_0^{\bar{\psi}} \varepsilon^2 \left( \frac{\partial \hat{F}_1}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_1}{\partial \zeta} \beta'_\xi \right) d\psi \right| \\ & \leq \left| \sum_{i=0}^k \int_{\psi_{i-1}}^{\psi_i} \varepsilon^2 \left( \frac{\partial \hat{F}_1}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_1}{\partial \zeta} \beta'_\xi \right) d\psi \right. \\ & \quad \left. + \int_{\psi_k}^{\bar{\psi}} \varepsilon^2 \left( \frac{\partial \hat{F}_1}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_1}{\partial \zeta} \beta'_\xi \right) d\psi \right| \\ & \leq \frac{C(\tau_0, C_1, C_2, C_3)}{\varepsilon} \|\Delta\xi\|_{C^0} \left| \sum_{i=0}^k \int_{\psi_{i-1}}^{\psi_i} \varepsilon^2 \left( \frac{\partial \hat{F}_1}{\partial \phi} \frac{\partial \phi}{\partial \psi} + \frac{\partial \hat{F}_1}{\partial \zeta} \frac{\partial^2 \phi}{\partial \psi^2} \right) d\psi \right| \\ & \quad + C(\tau_0, C_1, C_2, C_3)\varepsilon \|\Delta\xi\|_{C^0} \\ & \leq C(\tau_0, C_1, C_2, C_3)\varepsilon \|\Delta\xi\|_{C^0} \left| \sum_{i=0}^k (\hat{F}_1(\psi_{i+1}) - \hat{F}_1(\psi_i)) \right| \\ & \quad + C(\tau_0, C_1, C_2, C_3)\varepsilon \|\Delta\xi\|_{C^0} \\ & \leq C(\tau_0, C_1, C_2, C_3)\varepsilon \|\Delta\xi\|_{C^0}.\end{aligned}$$

It is easy to see

$$\begin{aligned}
& \left| \int_0^{\bar{\psi}} (\varepsilon^3 (\beta_\xi \phi_\psi + \phi \beta'_\xi) \mu + \varepsilon (\frac{\partial \hat{F}_2}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_2}{\partial \zeta} \beta'_\xi) \xi \right. \\
& \quad \left. + \varepsilon^3 \omega (\frac{\partial \hat{F}_3}{\partial \phi} \beta_\xi + \frac{\partial \hat{F}_3}{\partial \zeta} \beta'_\xi) d\psi \right| \\
& \leq C(\tau_0, C_1, C_2, C_3) \varepsilon \|\Delta \xi\|_{C^0}.
\end{aligned}$$

The last term is

$$\int_0^{\bar{\psi}} \varepsilon \hat{F}_2 \Delta \xi d\psi = \varepsilon \int_0^{\bar{\psi}} \phi (\frac{\partial \phi}{\partial \psi}) F_4(\phi, \phi_\psi) \Delta \xi d\psi.$$

From Lemma 3.5, by using the argument of the proof of Lemma 3.3, also using Corollary 3.12, we can prove

$$\left| \int_0^{\bar{\psi}} \varepsilon \hat{F}_2 \Delta \xi d\psi \right| \leq C(\tau_0, C_1, C_2, C_3) \varepsilon \|\Delta \xi\|_{C^1_{x_0}}.$$

So we have

$$|\beta_\xi(\psi_i)| \leq C(\tau_0, C_1, C_2, C_3) \varepsilon \|\Delta \xi\|_{C^1_{x_0}}. \quad (68)$$

Once we prove that

$$|\beta'_\xi(\psi_i)| \leq C(\tau_0, C_1, C_2, C_3) \|\Delta \xi\|_{C^1_{x_0}},$$

we can deduce (38) from (67).

Note that

$$\begin{aligned}
& |\beta'_\xi(\psi_{i+1}) - (\beta_\xi(\psi_i) \beta'_{1,i}(\psi_{i+1}) + \beta'_\xi(\psi_i) \beta'_{2,i}(\psi_{i+1}))| \\
& = \left| \int_{\psi_i}^{\psi_{i+1}} R_i^{-1} (\beta'_{2,i}(\psi) \beta_{1,i}(t) - \beta'_{1,i}(\psi) \beta_{2,i}(t)) \mathcal{L}_{\xi, \mu, \omega, \tau(0)} \beta_\xi(t) dt \right| \\
& \leq C(\tau_0, C_1, C_2, C_3) \varepsilon \|\Delta \xi\|_{C^0}
\end{aligned} \quad (69)$$

where

$$R_i(\psi) = \begin{vmatrix} \beta_{1,i}(\psi) & \beta_{2,i}(\psi) \\ \beta'_{1,i}(\psi) & \beta'_{2,i}(\psi) \end{vmatrix}$$

is the corresponding Wronskian. From Lemma 3.13 and (36) we can deduce

$$|\beta'_\xi(\psi_{i+1}) - \beta'_\xi(\psi_i)| \leq C(\tau_0, C_1, C_2, C_3) \varepsilon (\|\Delta \xi\|_{C^1_{x_0}} + \varepsilon \beta'_\xi(\psi_i)).$$

From  $\beta'_\xi(0) = 0$ , by an induction argument, we get

$$|\beta'_\xi(\psi_i)| \leq C(\tau_0, C_1, C_2, C_3) \|\Delta \xi\|_{C^1_{x_0}}.$$

So we have

$$\|\beta_\xi(\psi)\|_{C^1_\varepsilon} \leq C(\tau_0, C_1, C_2, C_3) \|\Delta \xi\|_{C^1_{x_0}}.$$

**3.  $\beta_\omega$  estimate** From

$$\frac{d}{dt}\phi_{\xi,\mu,\omega+t,\tau(0)}(\psi)|_{t=0} = \beta_\omega(\psi)$$

in the same way as before we can get (39).

**4. Proof of (40)**

$$\begin{aligned} & \frac{d}{dt}\tau(\bar{\psi}) \\ = & \frac{d}{dt}\int_0^{\bar{\psi}} \phi\phi_\psi\rho d\psi \\ = & \int_0^{\bar{\psi}} \varepsilon^2\left(\frac{\partial\hat{F}_1}{\partial\phi}\beta_\omega + \frac{\partial\hat{F}_1}{\partial\phi_\psi}\beta'_\omega\right) + \varepsilon^3(\beta_\omega\phi_\psi + \phi\beta'_\omega)\mu \\ & + \varepsilon\left(\frac{\partial\hat{F}_2}{\partial\phi}\beta_\omega + \frac{\partial\hat{F}_2}{\partial\phi_\psi}\beta'_\omega\right)R(\xi) + \varepsilon^3\omega\left(\frac{\partial\hat{F}_3}{\partial\phi}\beta_\omega + \frac{\partial\hat{F}_3}{\partial\phi_\psi}\beta'_\omega\right) + \varepsilon^3\phi_\psi^2 d\psi \end{aligned}$$

In the similar way as we did for  $\xi$ , we can get

$$\begin{aligned} & \left| \int_0^{\bar{\psi}} \varepsilon^2\left(\frac{\partial\hat{F}_1}{\partial\phi}\beta_\omega + \frac{\partial\hat{F}_1}{\partial\phi_\psi}\beta'_\omega\right) + \varepsilon^3(\beta_\omega\phi_\psi + \phi\beta'_\omega)\mu \right. \\ & \quad \left. + \varepsilon\left(\frac{\partial\hat{F}_2}{\partial\phi}\beta_\omega + \frac{\partial\hat{F}_2}{\partial\phi_\psi}\beta'_\omega\right)R(\xi) + \varepsilon^3\omega\left(\frac{\partial\hat{F}_3}{\partial\phi}\beta_\omega + \frac{\partial\hat{F}_3}{\partial\phi_\psi}\beta'_\omega\right) \right| \\ \leq & C(\tau_0, C_1, C_2, C_3)\varepsilon^3 \end{aligned}$$

The dominant term turns out to be

$$\int_0^{\bar{\psi}} \varepsilon^3\phi_\psi^2 d\psi.$$

So for  $(\xi, \mu, \omega, \phi(0))$  satisfy (21), we can choose  $\varepsilon$  sufficiently small, such that there is a uniform constant  $C_5 = C_5(\tau_0) > 0$ , which does not depend on  $C_1, C_2, C_3, \varepsilon$  such that

$$\frac{\partial}{\partial\omega}\tau\left(\frac{L_\Gamma}{\varepsilon}\right) \geq C_5(\tau_0)\varepsilon^2. \quad (70)$$

If  $\phi(0) = \frac{1-\sqrt{1-4\tau(0)}}{2}$  to  $\frac{1-\sqrt{1-4\tau(0)}}{2} + t$ , the linearized function is just  $\beta_1(\psi)$ .

$$\begin{aligned} & \left| \frac{\partial}{\partial\phi(0)}\left(\tau\left(\frac{L_\Gamma}{\varepsilon}\right) - \tau(0)\right) \right| \\ = & \left| \int_0^{\frac{L_\Gamma}{\varepsilon}} \varepsilon^2\left(\frac{\partial\hat{F}_1}{\partial\phi}\beta_1 + \frac{\partial\hat{F}_1}{\partial\zeta}\beta'_1\right) + \varepsilon^3(\phi_\psi\beta_1 + \phi\beta'_1)\mu \right. \\ & \quad \left. + \varepsilon\left(\frac{\partial\hat{F}_2}{\partial\phi}\beta_1 + \frac{\partial\hat{F}_2}{\partial\zeta}\beta'_1\right)R(\xi) + \varepsilon^3\omega\left(\frac{\partial\hat{F}_3}{\partial\phi}\beta_1 + \frac{\partial\hat{F}_3}{\partial\zeta}\beta'_1\right)d\psi \right| \\ \leq & K_1(\tau_0, C_1, C_2, C_3)\varepsilon \end{aligned}$$

where we deal with  $\varepsilon^2(\frac{\partial \hat{F}_1}{\partial \phi} \beta_1 + \frac{\partial \hat{F}_1}{\partial \phi_\psi} \beta'_1)$  in the same way as we did for  $\beta_\xi$  (note that we have Lemma 3.14, 3.15).

The estimates for  $|\frac{\partial \zeta(\frac{L\Gamma}{\varepsilon})}{\partial \omega}|$ ,  $\frac{\partial \zeta(\frac{L\Gamma}{\varepsilon})}{\partial \phi(0)}$  can be proved from Lemma 3.14, 3.15 and (39). At last we proved (40).

## References

- [1] R. Bettiol and P. Piccione. Delaunay type hypersurfaces in cohomogeneity one manifolds. *arXiv:1306.6043*, 2013.
- [2] C. Delaunay. Sur la surface de révolution dont la courbure moyenne est constante. *J. Math. Pure. Appl.*, 6:309–320, 1841.
- [3] J. Eells. The surfaces of delaunay. *Math. Intell.*, pages 53–57, 1987.
- [4] L.H. Huang. Foliations by stable spheres with constant mean curvature for isolated systems with general asymptotics. *Comm. Math. Phys.*, 300(2):331–373, 2010.
- [5] G. Huisken and S.T. Yau. Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature. *Invent. Math.*, 124:281–311, 1996.
- [6] N. Korevaar, R. Kusner, and B.Solomon. The structure of complete embedded surfaces with constant mean curvature. *J. Differ. Geom.*, 30(2):465–503, 1989.
- [7] F. Mahmoudi, R. Mazzeo, and F. Pacard. Constant mean curvature hypersurfaces condensing along a submanifold. *Geom. Funct. Anal.*, 16(4):924–958, 2006.
- [8] R. Mazzeo and F. Pacard. Constant mean curvature surfaces with delaunay ends. *Commun. Anal. Geom.*, 9(1):169–237, 2001.
- [9] R. Mazzeo and F. Pacard. Foliations by constant mean curvature tubes. *Commun. Anal. Geom.*, 13(4):633–670, 2005.
- [10] R. Mazzeo and F. Pacard. Constant curvature foliations in asymptotically hyperbolic spaces. *Rev. Mat. Iberoam.*, 27(1):303–333, 2011.
- [11] C. Nerz. Foliation by spheres with constant expansion for isolated systems without asymptotic symmetry. *arXiv:1501.02197v1*, 2015.
- [12] A. Neves and G. Tian. Existence and uniqueness of constant mean curvature foliation of asymptotically hyperbolic 3-manifolds. *Geom. Funct. Anal.*, 19(3):910–942, 12 2009.

- [13] A. Neves and G. Tian. Existence and uniqueness of constant mean curvature foliation of asymptotically hyperbolic 3-manifolds ii. *J. Reine. Angew. Math.*, 641:69–93, 2010.
- [14] F. Pacard and X. Xu. Constant mean curvature spheres in riemannian manifolds. *Manuscripta Mathematica.*, 128(3):175–295, 2009.
- [15] R. Rigger. The foliation of asymptotically hyperbolic manifolds by surfaces of constant mean curvature (including the evolution equations and estimates). *Manuscripta Math.*, 113(4):403–421, 2004.
- [16] R. Schoen. Uniqueness, symmetry and embeddedness of minimal surfaces. *J. Differ. Geom.*, 18:791–809, 1983.
- [17] R. Ye. Foliation by constant mean curvature spheres. *Pac. J. Math.*, 147(2):381–396, 1991.